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Reference

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ABSTRACT

Two important wood properties are the modulus of elasticity (MOE) and the modulus of rupture (MOR). In the past, the statistical distribution of the MOE has often been modeled as Gaussian, and that of the MOR as lognormal or as a two- or three-parameter Weibull distribution. It is well known that MOE and MOR are positively correlated. To model the simultaneous behavior of MOE and MOR for the purposes of wood system reliability calculations, we introduce a bivariate Gaussian–Weibull distribution and the associated univariate pseudo-truncated Weibull (PTW). We note that theoretical arguments suggest that the strength distributions of grades of lumber are likely to be PTW rather than Weibull. We describe a Web-based program that fits bivariate Gaussian–Weibull data sets (and thus fits PTW distributions to MOR data). We present data that demonstrate that strength distributions of visual grades of lumber are not Weibull and do display at least some of the characteristics of PTW data. Finally, we demonstrate via simulation that if we fit a Weibull distribution to PTW data (as is often done), we can obtain very poor estimates of probabilities of failure.

Keywords

wood system reliability, modulus of rupture, modulus of elasticity, Gaussian copula

Introduction

Two important wood properties are the modulus of elasticity (MOE) and the modulus of rupture (MOR). In the past, the statistical distribution of the MOE has often been modeled as Gaussian, and that of MOR as lognormal or as a two- or three-parameter Weibull distribution (see, for example, ASTM D2915 [1] and Refs 2 and 3).

Design engineers must ensure that the stresses to which wood systems are subjected rarely exceed the systems' strengths. To this end, ASTM D2915 [1], ASTM D245 [4], and ASTM D1990 [5] describe the manner in which "allowable properties" are assigned to populations of structural lumber. In essence, an allowable strength property is calculated by estimating a fifth percentile of a population (actually a 95 % content, one-sided lower 75 % tolerance bound) and then dividing that value by duration of load and safety factors. The intent is that the population can be used only in applications in which the stress does not exceed the allowable property. Of course there are stochastic issues associated with variable loads, uncertainty in estimation, and the division of a percentile with no consideration of population variability. Thus, this is not an ideal approach for ensuring the reliability of wood systems. However, it is the currently codified approach.

To apply this approach, one must obtain estimates of the fifth percentiles of MOR distributions. Currently, one method for obtaining estimates involves fitting a two-parameter Weibull distribution to a sample of MORs. To obtain this fit, either a maximum likelihood approach or a linear regression approach based on order statistics is permitted under ASTM D5457 [6].

These methods are often applied to populations that are not really distributed as two-parameter Weibulls. For example, in the United States, construction-grade 2 by 4 boards are often classified into visual categories (e.g., Select Structural, No. 1, No. 2) or into machine stress-rated (MSR) grades. In the case of MSR grades, MOE boundaries are selected, the MOE is measured non-destructively, and a piece of lumber is placed into a particular MSR category ("bin" in this paper) if its measured MOE lies between the MOE boundaries associated with that category. Because MOE and MOR are correlated, bins with higher MOE boundaries also tend to contain lumber populations with higher MOR values. The fifth percentiles of these MOR populations are sometimes estimated by fitting Weibull distributions to these populations. Upon reflection, statisticians and reliability engineers should recognize that this poses a problem. Even if the full population of lumber strengths were distributed as a Weibull, we would not expect that subpopulations formed on the basis of visual grades or MOE binning would continue to be distributed as Weibulls.

In fact, such a subpopulation is not distributed as a Weibull. If the full joint MOE–MOR population were distributed as a bivariate Gaussian–Weibull, the MOR subpopulation produced by MOE binning would be distributed as a univariate "pseudo-truncated Weibull" (PTW). In Ref 7, we obtained the distribution of a PTW and showed how to obtain estimates of its parameters and its quantiles by fitting a bivariate Gaussian–Weibull to the full MOE–MOR distribution. In the second section of this paper, we provide the density of our version of a bivariate Gaussian–Weibull and the densities of the associated truncated bivariate Gaussian–Weibull and univariate PTW.

In the third section we describe a Web-based computer program that we have developed that fits bivariate Gaussian–Weibull distributions to bivariate data. In the fourth section we demonstrate that real MOR data do not follow a two-parameter Weibull distribution and do have at least some of the characteristics that we would expect from PTW data. Finally, in the fifth section we present results from a simulation in which we investigated how poorly we can do in estimating failure probabilities when we fit Weibull distributions to PTW data.

We note that the bivariate Gaussian–Weibull distribution has uses other than as a generator of PTW distributions. For example, engineers who are interested in simulating the performance of wood systems must begin with a model for the joint stiffness and strength distribution of the members of the system (see, for example, Refs 8–10). Provided that we are considering the *full* population, a Gaussian–Weibull is one possible model for this joint distribution.

Prior to the publication of Ref 7, bivariate Gaussian–Weibull distributions had not yet appeared in the literature. However, bivariate Weibull distributions have previously been investigated (see, for example, Refs 11–23).

We note that the bivariate Gaussian–Weibull distribution that we discuss in the current paper is not the only possible bivariate distribution with Gaussian and Weibull marginals. In essence, we begin with a "Gaussian copula," a bivariate uniform distribution generated by starting with a bivariate normal distribution and then applying normal cumulative distribution functions to its marginals. (Reference 7 follows Ref 24 in its development. References 25 and 26 introduced this technique in a lumber context.) However, there is a large body of literature on alternative copulas (multivariate distributions with uniform marginals); see, for example, Refs 27 and 28. These alternatives would lead to alternative bivariate Gaussian–Weibulls. Ultimately, the test of the usefulness of our proposed version of a Gaussian–Weibull for a particular application will depend on the match between the theoretical distribution and data. Still, we believe that the analysis of our proposed version in the current paper and in Refs 7, 29, and 30 represents a useful step in the construction and evaluation of bivariate Gaussian–Weibull distributions.

Densities

A BIVARIATE GAUSSIAN–WEIBULL DISTRIBUTION

The density function of the version of a bivariate Gaussian–Weibull distribution introduced in Ref 7 is

$$\begin{aligned}&\text{gaussweib}(x, w; \mu, \sigma, \rho, \gamma, \beta)\\ &\quad \equiv \gamma^{\beta}\beta w^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right) \times \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma\sqrt{1-\rho^2}}\\ &\quad \times \exp\left(-\frac{\left(\frac{x-\mu}{\sigma}-\rho y\right)^2}{2(1-\rho^2)}\right)\end{aligned} \tag{1}$$

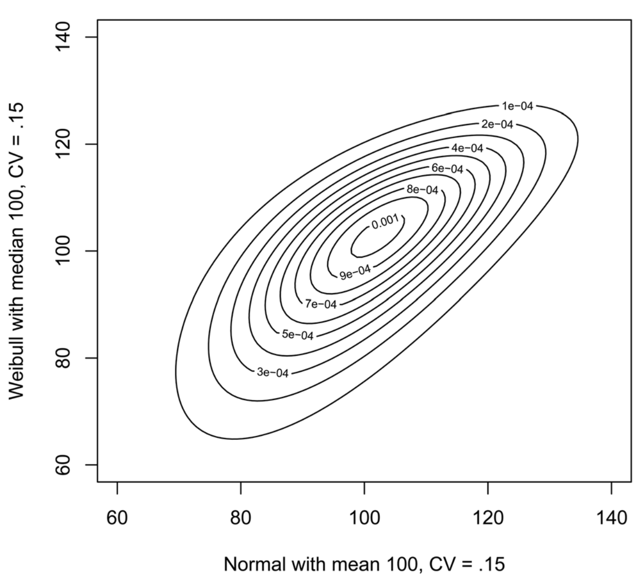
where:

$$y = \Phi^{-1}\left(1-\exp\left(-(\gamma \times w)^{\beta}\right)\right)$$

Φ is the $N(0,1)$ cumulative distribution function, and μ, σ, ρ, γ, and β are the parameters of the distribution.

In **Fig. 1** we provide a contour plot of the bivariate Gaussian–Weibull distribution for a coefficient of variation (CV) equal to 0.15 and a "generating correlation" (ρ) equal to 0.7. (In Appendix B of Ref 31, we report simulations that indicate that the sample correlation between the marginal Gaussian and Weibull random variables will be very close to the generating correlation.) Additional plots are provided in Ref 7. Note in these plots that as the CV declines from 0.35 to 0.25 to 0.15 (as the Weibull shape parameter increases from 3.13 to 4.54 to 7.91), the density contours become much less elliptical. That is, the distribution diverges from bivariate normal. We would expect this, as a Weibull distribution is "like a normal" for shape near 3.6 (skewness equals 0.00056, excess kurtosis equals −0.28), and a Weibull becomes skewed to the left and leptokurtic as the shape increases.

FIG. 1 Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.15 and a generating correlation of 0.7.



A TRUNCATED BIVARIATE GAUSSIAN–WEIBULL DISTRIBUTION

In wood engineering applications, it is often the case that we do not have data from a full bivariate Gaussian–Weibull distribution. Instead, we have data from the subpopulation that is formed by considering lumber whose MOE values lie between two pre-determined limits, c_l and c_u (that is, we have MSR lumber). It is clear that the joint density in this case is

$$\text{gaussweib}(x, w; \mu, \sigma, \rho, \gamma, \beta)\Big/\left(\Phi\left(\frac{c_u-\mu}{\sigma}\right)-\Phi\left(\frac{c_l-\mu}{\sigma}\right)\right) \tag{2}$$

for x between c_l and c_u, and 0 elsewhere.

THE PSEUDO-TRUNCATED WEIBULL DISTRIBUTION

In Ref 7 we show that the density function for a PTW (based on a bivariate Gaussian–Weibull) is

$$\begin{aligned}f_{\text{PTW}}(w) &= \gamma^{\beta}\beta w^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\\ &\quad \times\left(\Phi\left((c_u-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}\right)\right.\\ &\quad \left.-\Phi\left((c_l-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}\right)\right)\Big/\\ &\quad \left(\Phi\left(\frac{c_u-\mu}{\sigma}\right)-\Phi\left(\frac{c_l-\mu}{\sigma}\right)\right)\end{aligned} \tag{3}$$

where

$$y = \Phi^{-1}\left(1-\exp\left(-(\gamma w)^{\beta}\right)\right)$$

Thus, as we would expect, for $\rho = 0$, the PTW density is simply the Weibull density, $\gamma^{\beta}\beta w^{\beta-1}\exp(-(\gamma w)^{\beta})$.

In Appendix K of Ref 7, we show that as $\rho \to 1$, the density of a PTW distribution converges to the density of a truncated Weibull distribution.

Note that the PTW density in Eq 3 nominally involves seven parameters ($\mu, \sigma, \rho, \beta, \gamma, c_l$, and c_u). In fact, however, it is a five-parameter density. To see this, we define

$$p_l \equiv \Phi\left(\frac{c_l-\mu}{\sigma}\right)$$

and

$$p_u \equiv \Phi\left(\frac{c_u-\mu}{\sigma}\right)$$

so that $p_u - p_l$ is the probability that the Gaussian variable lies between the truncation limits c_l and c_u. Then we can replace $(c_l-\mu)/\sigma$ and $(c_u-\mu)/\sigma$ in Eq 3 with $\Phi^{-1}(p_l)$ and $\Phi^{-1}(p_u)$, and we see that the density is actually a function of the five parameters ρ, β, γ, p_l, and p_u. Further, in some practical situations,

FIG. 2 Weibull probability plot of a pseudo-truncated Weibull distribution with generating coefficient of variation equal to 0.25 and generating correlation equal to 0.0. The straight line is the ordinate-equals-abscissa line.

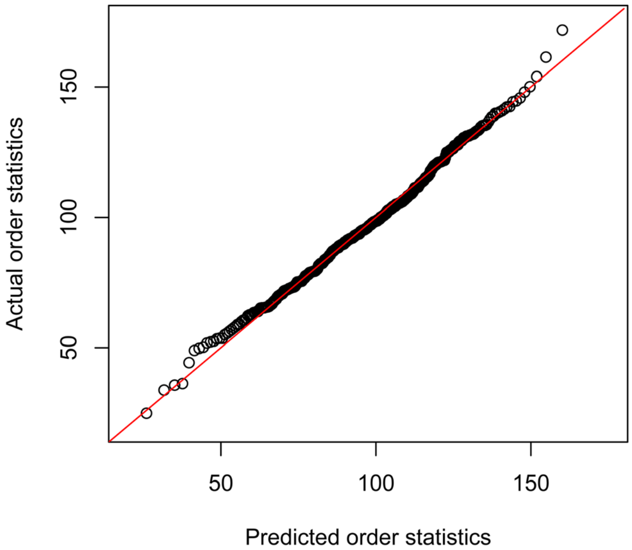
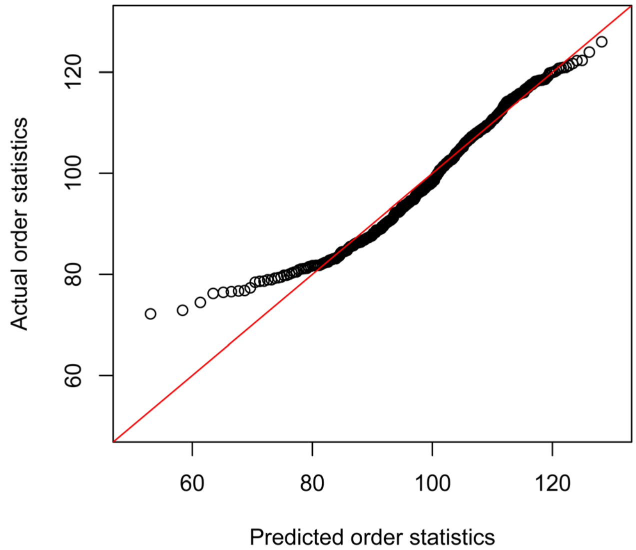


FIG. 3 Weibull probability plot of a pseudo-truncated Weibull distribution with generating coefficient of variation equal to 0.25 and generating correlation equal to 0.99. The straight line is the ordinate-equals-abscissa line.



we can take p_l and p_u as known. For example, historical data might permit us to conclude that for No. 2 lumber from a particular mill or region, $p_l = 0.4$ and $p_u = 0.8$ (i.e., 40 % of the lumber produced by the mill or region is graded as poorer than No. 2, and 20 % of the lumber is graded as better than No. 2). This would simplify the task of fitting a PTW distribution to MOR data in the absence of correlated MOE data or truncation bounds on the MOE data (as would be available for MSR data).

Figures 2 and **3** are (one version of) Weibull probability plots of PTW data. We plotted the ordered data from a PTW sample against the predicted ordered data from the best Weibull fit to the data. If the data really were Weibull, then the plots would be approximately linear. In **Fig. 2**, the generating X, Y correlation was 0, so the data actually were Weibull and the plot is approximately linear. In **Fig. 3**, the generating X, Y correlation was 0.99, so the data were "far from Weibull" and the plot is quite nonlinear. For both data sets, the Weibull coefficient of variation was 0.25 and c_l and c_u corresponded to the 0.2 and 0.8 quantiles of the Gaussian distribution.

In Appendix L of Ref 7, we formally establish that for $\rho \neq 0$, PTW distributions are not Weibull distributions.

Web Program to Estimate the Parameters of a Bivariate Gaussian–Weibull

Based on the theory in Ref 7, we have developed a computer program that obtains asymptotically efficient estimates of the parameters of a bivariate Gaussian–Weibull distribution. The program also performs Anderson–Darling and Cramér–von Mises Gaussian and Weibull goodness-of-fit tests of the marginal distributions; returns nominal 75 %, 90 %, 95 %, and 99 % confidence intervals on the parameters; and performs simulations to obtain estimates of the actual coverages of the confidence intervals. The program's user interface is described in Section 6 of Ref 29. Algorithmic details of the program are provided in Appendix B of Ref 29. The Web program can be run at http://www1.fpl.fs.fed.us/fit_gauss_weib.html. The code for a standalone FORTRAN program that performs these same functions can be found at http://www1.fpl.fs.fed.us/fit_gauss_weib_ code.html.

We are currently developing software that fits truncated (on the Gaussian) bivariate Gaussian–Weibull data (that is, MSR data). It has performed well in simulations and will be described in a technical report.

We have also investigated software that fits pure PTW data. As explained in the section "The Pseudo-truncated Weibull Distribution," these are univariate data (for example, data in which only the MOR is available) that are parameterized by μ, σ, ρ, p_l, and p_u. We have found that it is numerically difficult to obtain good estimates in this case, especially when p_l does not approach 0 and p_u does not approach 1 (as in the case of No. 2 lumber). However, we have been having some success when subsets of the parameters (e.g., p_l and p_u) can be taken as known. This work continues.

Empirically, Are Strength Populations of "Bins" Pseudo-truncated Weibull Rather than Weibull?

We do have evidence that strength populations of bins are not Weibull. For example, in **Table 1** we list 19 cells of Select Structural and No. 2 data that were obtained in the Ingrade Program (Refs 2 and 3). In this table, DF, SP, and HF denote, respectively, Douglas Fir, Southern Pine, and Hem Fir. We also present significance levels from Cramér–von Mises (CVM) and Anderson–Darling (AD) tests of goodness of fit (see, for example, Ref 32) of a two-parameter Weibull distribution. In most cases, a two-parameter Weibull distribution is rejected by both tests.

We also produced Weibull probability plots of the 19 data sets. **Figure 4** is an example of these plots. (In our "probability plots," we plot ordered data versus predicted ordered data.) Note the short left tail and the concave down right tail. Sixteen of the 19 plots had a short left tail. None had a long left tail. Thirteen of the 19 plots had a concave down right tail. Only one had a concave up right tail. (This appearance conforms with that of Weibull probability plots of generated PTW data; see, for example, **Fig. 3**.)

Thus, as expected from the theoretical treatment in Ref 7, we can conclude that *strength populations created via visual grading of lumber are not Weibull distributed*. However, do they have our PTW distribution? Unfortunately, it is not currently possible to answer this question. It has not been the practice to collect full population data. Instead, lumber has been sorted into, for example, visual categories (e.g., Select Structural, No. 1, No. 2), and for a given population, measurements have been made on only some of these categories. We have not been able to obtain paired MOE and MOR values for a random sample from a full population of lumber.

TABLE 1 Cramér–von Mises and Anderson–Darling goodness-of-fit tests for ingrade data [2,3].

				Goodness-of-fit p Value	
Species	Lumber Size	Grade[a]	Sample Size	CVM	AD
DF	2 × 4	SS	414	0.10	0.05
DF	2 × 8	SS	493	NS	NS
DF	2 × 10	SS	414	NS	NS
DF	2 × 4	2	386	0.25	0.10
DF	2 × 8	2	1964	0.01	0.01
DF	2 × 10	2	388	0.01	0.01
HF	2 × 4	SS	428	0.05	0.01
HF	2 × 8	SS	375	0.05	0.05
HF	2 × 10	SS	368	0.10	0.05
HF	2 × 4	2	406	0.01	0.01
HF	2 × 8	2	372	0.01	0.01
HF	2 × 10	2	361	0.01	0.01
SP	2 × 4	SS	413	0.05	0.025
SP	2 × 8	SS	626	0.05	0.025
SP	2 × 10	SS	413	0.01	0.01
SP	2 × 4	2	413	0.01	0.01
SP	2 × 6	2	413	0.01	0.01
SP	2 × 8	2	1367	0.01	0.01
SP	2 × 10	2	412	0.10	0.25

[a]SS, select structural.

Also, we cannot treat such data as bivariate Gaussian–Weibull data that have had their Gaussian component truncated. In contrast to MSR lumber, visually graded lumber is binned according to (informed) judgments made by a human grader. There is, perhaps, an implicit normally distributed visual variable by which the material has been binned. However, we do not have values for that variable.

We are working to obtain paired MOE/MOR values for either full populations or MSR lumber. This should permit us to perform goodness-of-fit tests.

However, regardless of whether our particular form of a bivariate Gaussian–Weibull distribution is optimal for stiffness-strength lumber data, we believe that the analysis in the next section should lead to caution in the use of the Weibull distribution to model PTW data.

If Strength Populations Are Pseudo-truncated Weibull Rather than Weibull, Does It Matter?

We have discussed how the joint distribution of the full population of wood stiffness and strength can be modeled as bivariate Gaussian–Weibull. As noted in the Introduction, a bivariate Gaussian–Weibull MOE–MOR joint distribution would lead to MSR bins and visual grades of lumber that have PTW strength

FIG. 4 Weibull probability plot for Hem Fir 2 × 4, number 2 lumber.

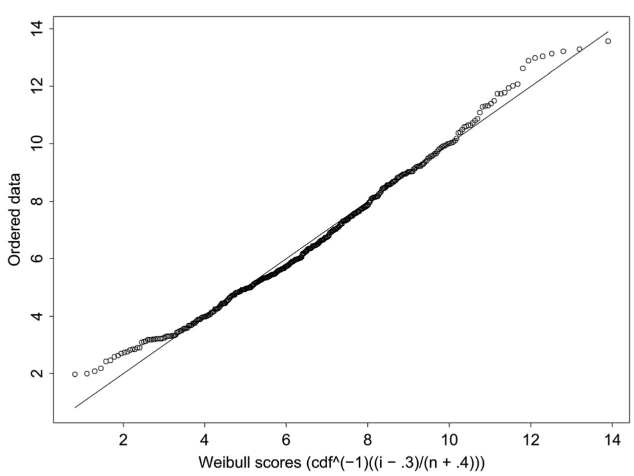


TABLE 2 Frequencies for $p_{F,true}/p_{F,est}$ for a bivariate Gaussian–Weibull distribution with generating correlation = 0.70 and CV = 0.30.

Fit	Median p_F Ratio	p_F Ratio Frequencies						
		$[0, 0.02]$	$(0.02, 0.2]$	$(0.2, 0.5]$	$(0.5, 2)$	$[2, 5)$	$[5, 50)$	$[50, \infty)$
W/Reg/All	0.41	0.0000	0.0261[a,b]	0.6829[a,b]	0.2910[a,b]	0.0000	0.0000	0.0000
W/Reg/20	0.73	0.0000	0.0364[a,b]	0.2766[a,b]	0.5670[a,b]	0.1018[b]	0.0182[b]	0.0000
W/Reg/10	0.79	0.0000	0.0548[a,b]	0.2664[a,b]	0.4771[a,b]	0.1383[b]	0.0617[a,b]	0.0017[a,b]
W/MLE/All	0.27	0.0000	0.1854[a,b]	0.7907[a,b]	0.0239[a,b]	0.0000	0.0000	0.0000
W/MLE/20	0.74	0.0000	0.0144[a,b]	0.2540[a,b]	0.6560[a,b]	0.0697[b]	0.0059[b]	0.0000
W/MLE/10	0.92	0.0001[a,b]	0.0187[a,b]	0.2069[a,b]	0.5709[a,b]	0.1549[a,b]	0.0479[a,b]	0.0006[a,b]
PTW cens	1.04	0.0000	0.0005[b]	0.1047[b]	0.7051[b]	0.1536[b]	0.0360[b]	0.0001[b]
PTW 400	1.03	0.0000	0.0000	0.0343	0.9090	0.0567	0.0000	0.0000
PTW 1000	1.01	0.0000	0.0000	0.0023	0.9940	0.0037	0.0000	0.0000

Notes: p_F denotes the probability of failure when a stress equal to the allowable bending strength is applied. "W" in the "Fit" column indicates that the fit was made on the basis of an (incorrect) Weibull assumption. "PTW" indicates that the fit was made on the basis of a (correct) PTW (or bivariate Gaussian–Weibull) assumption.
[a]"Worse" behavior than PTW cens behavior.
[b]"Worse" behavior than PTW 400 behavior.

distributions. It is often current practice to fit Weibull distributions to these subpopulations. The fits have been based on both regression and maximum likelihood approaches, and on censored and uncensored data [6]. (The censoring has been justified by wood scientists by the argument that they are only concerned about lower tail behavior and by their belief that they have seen "better fits" to lower tails via censored Weibull fits than full Weibull fits; see, for example, Section X.1.1 of Ref 6. For statisticians and reliability engineers, this rings alarm bells. Throwing away data increases variance, and using the censored Weibull, in essence, as an empirical "spline" for the near left tail does not give any assurance that it will predict well in the far tail.) "Allowable bending strengths" are then calculated (essentially) as fifth percentiles divided by a 2.1 duration of load and safety factor. We were interested in the effect of these practices on the estimation of probabilities of failure when stresses are equal to allowable bending strengths.

We conducted simulations to investigate this question. (A listing of the simulation program can be found at http://www1.fpl.fs.fed.us/bivar3.html; see the pfsim4.f link.)

For four generating correlations (0.50, 0.60, 0.70, and 0.80) and three CV values (20, 30, and 40), we generated 10 000 bivariate Gaussian–Weibull data sets. To simulate number 2 lumber, these data sets were generated so that each yielded 400 specimens with a Gaussian value that lay between the 40th and 80th percentiles of the Gaussian distribution. This required approximately 1000 bivariate Gaussian–Weibull pairs in each data set. For each of these 10 000 data sets, we calculated Weibull regression and maximum likelihood fits based on all 400 simulated pseudo-truncated MOR data values (W/Reg/All and W/MLE/All) and censored data fits based on the bottom 20 % (W/Reg/20 and W/MLE/20) and bottom 10 % (W/Reg/10 and W/MLE/10) of the values. We also obtained a bivariate Gaussian–Weibull "maximum likelihood" fit (a purely numeric optimization of density [3]) to the 400 data pairs binned by the Gaussian value (PTW cens), a bivariate Gaussian–Weibull fit (based on Theorem 1 in [7]) to the first (not the lowest) 400 of the approximately 1000 unbinned Gaussian–Weibull values (PTW 400), and a bivariate Gaussian–Weibull fit of all of the approximately 1000 unbinned Gaussian–Weibull values (PTW 1000).

Each of these fits yielded an allowable bending strength. In each case we calculated the predicted probability that the MOR would lie beneath the allowable bending strength. This value was calculated from the fitted distribution. In each case we also calculated the true probability that the MOR would lie beneath the allowable bending strength. This value was calculated from the (known) generating bivariate Gaussian–Weibull. Information about the ratios of the true "probabilities of failure" (probability that a specimen has strength lower than the allowable bending strength) to the estimated probabilities of failure is presented in Tables 6 through 9 of Ref 30. In **Table 2** of the current paper, we provide a condensed version of a portion of Ref 30's Table 8. This condensed version only contains results for a generating correlation equal to 0.70 and a CV equal to 0.30.

In column 1 of **Table 2**, we describe the type of fit. These values correspond to the nine fitting techniques described above. In column 2 we provide the median of the 10 000 ratios of true to estimated probabilities of failure. A value less than 1 indicates an approach that is conservative at the median. However, as can be seen from the remainder of the table, the median represents an insufficient summary. The remaining columns list the fraction of the time for which a particular technique had ratios of true to estimated probabilities of failure that lay in the intervals $[0, 0.02]$, $(0.02, 0.2]$, $(0.2, 0.5]$, $(0.5, 2)$, $[2, 5)$, $[5, 50)$, and $[50, \infty)$. Obviously, we could have chosen other intervals. However, our main points are clear from the table(s):

1. The Weibull fits (regression and maximum likelihood) tend to be overly conservative. That is, true probabilities of failure can be much less than estimated probabilities of failure.
2. At the same time, the censored (20 % and 10 %) Weibull fits can be much more variable than the correct bivariate Gaussian–Weibull fits, with the result that they can occasionally yield highly nonconservative fits. That is, the actual probabilities of failure can be much greater than the estimated probabilities of failure.
3. If the joint MOE/MOR distribution is truly bivariate Gaussian–Weibull, we can obtain better estimates of the probability of failure by taking the same number of specimens from the full distribution than by restricting ourselves to binned values. (Compare the PTW 400 and PTW cens results.)

Summary

In the context of wood strength modeling and reliability calculations, we have introduced a bivariate Gaussian–Weibull distribution and the associated univariate pseudo-truncated Weibull (PTW) distribution. We have described a Web-based program that fits bivariate Gaussian–Weibull data sets (and thus permits estimation of PTW distributions, provided c_l and c_u are known and bivariate Gaussian–Weibull data are available). In Ref 7, we demonstrate theoretically that if full MOE–MOR populations have bivariate Gaussian–Weibull distributions, then the MOR distributions of MSR grades of lumber will be PTW, and the MOR distributions of visual grades of lumber will be at least approximately PTW. In the current paper, we have demonstrated empirically that the MOR distributions of visual grades of lumber are not Weibull and that they have at least some of the characteristics of PTW populations. Finally, we have demonstrated that, as one would expect, if we fit Weibull distributions to PTW data, we can obtain very poor estimates of probabilities of failure. These results suggest that ASTM standards for estimating the allowable strength properties of lumber grades should not permit a Weibull assumption.

We note that in this paper, we are arguing not so much *for* a PTW model for strength data as *against* a Weibull one. A reviewer remarked that they had seen evidence that strength distributions associated with lumber grades might need to be modeled as mixtures of parametric distributions; we agree that this is likely to be true. In this case, the strength distribution of a grade of lumber might be modeled as a mixture of PTW distributions, and there would still be a "thinning of the tail" due to pseudo-truncation.

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