Determination of transverse elastic constants of wood using a cylindrically orthotropic model

John C. Hermanson\textsuperscript{1}, Suzanne M. Peyer\textsuperscript{2} and Jay A. Johnson\textsuperscript{3}

\textsuperscript{1} USDA Forest Products Laboratory, Madison WI, USA; \textsuperscript{2} University of Wisconsin-Madison, Madison WI, USA; \textsuperscript{3} University of Washington, Seattle WA, USA

Abstract

The arrangement of anatomical elements in the cross section of a tree can be characterized, at least to a first approximation, with a cylindrical coordinate system. It seems reasonable that the physical properties of wood in the transverse plane, therefore, would exhibit behaviour that is associated with this anatomical alignment. Most of the transverse properties of wood reported in the literature were determined using test specimens with rectilinear geometry. We are interested in using test specimens that are geometrically compatible with the material organization of the tree to avoid the boundary condition problems that arise from testing materials with curvilinear orthotropy. To this end, we are developing a test procedure to obtain values for the transverse elastic constants from hollow disks of wood subjected to various loading regimes. In this paper we show, for a given set of displacements and forces in two classes of problems: 1) hollow disk subjected to equally spaced radial loads and 2) hollow disk subjected to equally spaced tangential loads, that previously developed theoretical solutions match finite element analysis. We then discuss how the theoretical solutions can be used in an inverse method to determine the transverse elastic constants.

Introduction

The anatomical arrangement of plant cells in the trunk, or in the branch stems, of the tree suggests the use of a cylindrically orthotropic (CYO) elastic stress-strain law to characterize the mechanical behaviour of wood in the transverse plane (i.e., the plane perpendicular to the grain or longitudinal direction). The CYO stress-strain law is well known. See Lekinitskii [1] for example. The objective of this paper is to develop the methodology to test wood in its anatomical arrangement to calculate the transverse properties.

The transverse properties will be useful for simulations of drying stresses, in structural analysis such as small diameter timber structures and other structural applications that require the complete set of three dimensional elastic constants. We believe the transverse elastic constants may also be potential indicators of wood quality. We think that the lack of information about these constants represent a gap in the knowledge database of wood. To this end we are developing a new procedure to determine the elastic constants.

Typically, rectilinear specimens are used to determine the elastic constants even though the principle material directions of the wood within the specimen changes. These changes in principle material directions lead to non-homogeneity in the stress and strain fields, which leads to erroneous calculations of the elastic constants. This new procedure is based on hollow wood disk specimens that preserve the cylindrically orthotropic nature of the wood. This procedure involves using theoretical expressions containing the four independent transverse elastic constants and the load and displacement boundary conditions. The number of equations to solve for the four unknown elastic constants is a function of the number of tests and boundary measurements taken. The inverse method is used to determine what the elastic must be to satisfy the measured boundary conditions during the test. The
theoretical expressions are of two types: 1) equally spaced radial loads acting on the outer circumference of the disk (Fig 1) and 2) equally spaced tangential loads acting on the outer circumference of the disk (Fig 2). The inner circumferential surface of the disk is stress free in the radial regime but is not allowed to displace in the tangential loading regime. To verify whether this procedure is viable, we used finite element analysis of the loading states and calculated displacements. We then compared these displacements to those obtained via the analytical solution.

Material and methods

Theoretical expressions for the disk displacements are obtained from the solution for stresses in a CYO material. The general formulation for problems of this type has been given elsewhere \[22\]. This paper outlines how to use the general formulation of Johnson et al. \[2\] using the inverse method. The ingredients of the problem involved the CYO elastic stress-strain law in which the cylindrical strain components are related to the cylindrical stress components through the elastic constants: \(E_T, E_R, G_{RT}\) and \(V_{TR}\), where \(E_{\infty}, E_S\) Young’s moduli in the tangential and radial directions respectively, \(G_{RT}\) is the shear modulus in the radial-tangential plane and \(V_{TR}\) is Poisson’s ratio for the radial-tangential plane. We have found it convenient to use \(E_T\) as a scaling parameter and to replace the other three constants with three dimensionless parameters \(v, a\) and \(b\), where \(v = \sqrt{V_{TR}V_{RT}}, a = E_T/E_R\) and \(b = (E_T/2G_{RT}) - V_{TR}\). With these new parameters, the CYO stress-strain law (plane stress) can be written as:

\[
\begin{bmatrix}
\sigma_r \\
\sigma_\theta \\
\tau_{r\theta}
\end{bmatrix} = \frac{1}{E_T} \begin{bmatrix}
\frac{a}{2} - v^2 r_T \alpha & 0 & -v \sqrt{\alpha} \\
0 & 2[b + v \sqrt{\alpha}] & 0 \\
-v \sqrt{\alpha} & 0 & 1
\end{bmatrix} \begin{bmatrix}
\sigma_\alpha \\
\tau_{r\alpha} \\
\tau_{\alpha\alpha}
\end{bmatrix}
\]

The other ingredients of the problem are the definitions of the stress components in terms of a stress function and the strain compatibility equations. Combining these ingredients leads to a fourth order partial differential equation for the stress function. The two classes of problems used in this paper involved the specification of equally spaced loads around the circumference of the hollow wood disk. In the first class of problems, the equally spaced loads are radial and in the second they are tangential. We use dimensionless variables everywhere to compactly display the essential details of the problems. Thus the radial coordinate is expressed in terms of \(\xi = \xi_1\), where \(r_1\) is the outer radius of the disk. The inner radius, \(r_0\), is expressed as \(\xi = r/r_\infty\). The angular position is \(\theta\) in radians, which is dimensionless. The \(N\) equally spaced loads are expressed as Fourier expansions of a delta function which can be thought of as pressure (radial) or shear (tangential) distribution composed of a sum of sinusoidal loadings:

\[
\sigma = \frac{1}{\sqrt{\alpha}} \left( p_1 \cos(1,\theta)/q_1 + \sin(1,\theta)/q_1 \right)
\]

where: \(\sigma = \sigma(1,\theta)/p_1, \tau = \tau(1,\theta)/q_1, p_1 = NP/2\pi hr_1\) and \(NQ/2\pi hr_1\). Each of the radial (tangential) point loads is referred to as \(P\) (or \(Q\)) and the disk thickness is \(h\). In the first class of problems there are no surface tractions on the inner radial circumference of the disk and in the second the radial and tangential displacements are set to zero on the inner surface (i.e., the surface is clamped to prevent rotation). The pressure, or shear distribution suggests a Fourier expansion of the stress function with the coefficients of \(\cos(k\theta)\) and \(\sin(k\theta)\) being functions of the radial coordinate \(\xi\). After lengthy calculations it can be shown that the displacement functions take the form:

\[
U^* = (2\pi E_T/N)(u_1/P) \text{ (replace } P \\text{ with } Q \text{ and } u_1 \text{ with } u_\theta \text{ for the tangential load case)}
\]

The explicit form of the formulas for \(U^*_0\) and \(U^*_k\)
In the expression for $U^* u_r/P$ represents a compliance factor which can be determined through testing, i.e., $\lambda = \Delta u_r/\Delta P$. Thus, several normalized compliances factors, $A^* = (2\pi h E_T/N)\lambda$, can be computed using the expressions for $U^*$ with appropriate ($\xi, \theta$) coordinates and correct values of the dimensionless elastic parameters: $v, a, b$. Conceptually if several compliance values were determined from a mix of loading regimes ($N = 1, 2...N_1$ radial loads and $N - 1, 2...N_2$ tangential loads, measured at different locations on the wood disks) then the sum of square differences between the measured normalized compliances and the dimensionless displacements will be zero.

The idea behind the proposed procedure is to measure a set of compliances and use a minimization algorithm to find the values of the elastic parameters that will make the sum of the square differences a minimum.

\begin{equation}
\Delta \lambda = \Delta u_r/\Delta P.
\end{equation}

In this paper, we used finite elements to simulate the displacement states of wood disks under various loadings to obtain measured compliances. These will be compared with the displacements that were calculated in the finite element simulations. The ADINA finite element package was used in the simulations. Four test cases of elastic constants were considered under both radial and tangential loading, Table 1. Fig. 1, and Fig. 2 illustrate the radial and tangential load case respectively and Fig. 3 shows the mesh used in the finite element calculations. We used a finite element simulation of a hollow disk with an outer radius of $r_1 = 70.6$ mm and an inner radius of $r_0 = 12.7$ mm, thus $\xi_0 = 0.0899$. We used $N = 2$ radial and $N = 2$ tangential unit loads ($P = Q = 1$). The radial displacement $u_r$ was determined at two locations along the inner surface $\xi = \xi_0$ and $\theta = 0$ and $\pi/2$, for the radially loaded disk, Fig. 1. The tangential displacement $u_\theta$ was determined at two locations along the ray $\theta = \pi/2$ at $\xi_1 = 0.551$ $\xi_2 = 0.910$, Fig. 2.

The selection of these values for the geometric parameters was based on actual wood samples whose test results will be reported in a future publication. Furthermore, the minimization algorithm is still under development. Consequently, we will simply compare the theoretical (series solution) with the FE displacement results.

**Results and discussion**

For the purpose of comparison, all results have been normalized as formulated above in Equations (1) through (3). The normalized displacement results of the theoretical analysis, finite element analysis and their relative difference, $100(U^*_{\text{theory}} - U^*_{\text{FEM}})/U^*_{\text{theory}}$, are listed in Table 2 for each load (radial and tangential) and material (I-IV) case. The relative difference ranges from 4.0% to -0.2%. With the exception of the single occurrence of -0.2% all of the relative differences were greater than zero, which is to be expected from a finite element versus analytical comparison. Finite element analysis, with the isoparametric formulation used, overestimate the stiffness of the system yielding smaller displacements compared to exact analytical analysis. Nevertheless, the comparison between the analytical analysis and the finite element analysis are very close. We believe the results could be made closer by refining the mesh of the finite element analysis.

Having established, in this paper, the validity of the theoretical equations to calculate displacements given a set of point load boundary conditions, the next step will be to utilize Equation (4) in an inverse method, using as input the boundary conditions and results of the finite element analysis. The elastic constants from the inverse method will be compared to those input into the finite element analysis. The last step is to use the inverse method on data obtained from actual wood disks.
Fig. 1 A schematic of the radial load case with $N=2$ is depicted. $\xi$ is the non-dimensional radial coordinate, $\xi_0$ is the inner radius of the hollow disk. $\theta$ is taken as counterclockwise from the axis of the load. Displacements are measured on the inner radius, $\xi_0$, at $\theta=0$ and $\theta=\pi/2$.

Fig. 2 A schematic of the tangential load case with $N=2$ is depicted. Displacements are measured on the inner radius at $\xi_1=0.551$ and $\xi_2=0.910$ and $\theta=\pi/2$ as shown.

Fig. 3 The finite element mesh used to provide displacements for comparison to the theoretical analysis. The elements used were 8-node isoparametric. Symmetry for the radial and antisymmetry for the tangential was used to model the entire disk. The symbols in each element represent the principal material orientation at the centre of the element. The mesh shown illustrates a radial load condition. A tangential load at the same nodal point was also calculated with the material properties shown in Table 1.
Table 1. Input parameters a, b and \( v \) and the corresponding normalized elastic constants for the infinite element
analysis \( E_T = 1 \) and \( v_T = 0.4 \)

<table>
<thead>
<tr>
<th>Case</th>
<th>( a )</th>
<th>( b )</th>
<th>( v )</th>
<th>( E_T/E_R )</th>
<th>( G_{RT}/E_T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>0.5</td>
<td>2</td>
<td>0.283</td>
<td>2</td>
<td>0.227</td>
</tr>
<tr>
<td>II</td>
<td>0.5</td>
<td>6</td>
<td>0.283</td>
<td>2</td>
<td>0.081</td>
</tr>
<tr>
<td>III</td>
<td>2.0</td>
<td>2</td>
<td>0.566</td>
<td>5</td>
<td>0.179</td>
</tr>
<tr>
<td>IV</td>
<td>2.0</td>
<td>6</td>
<td>0.566</td>
<td>5</td>
<td>0.074</td>
</tr>
</tbody>
</table>

Table 2. Comparison of the results from the theoretical analysis and finite elements for the two loading, radial
and tangential, and four material cases.

<table>
<thead>
<tr>
<th>Case</th>
<th>Radial load ( P, N=2 )</th>
<th>Tangential load ( Q, N=2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( U^*(\xi,0) )</td>
<td>( U^*(\xi,\pi/2) )</td>
</tr>
<tr>
<td>I</td>
<td>Theory</td>
<td>2.06</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>2.03</td>
</tr>
<tr>
<td></td>
<td>( \Delta% )</td>
<td>1.49</td>
</tr>
<tr>
<td>II</td>
<td>Theory</td>
<td>5.73</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>5.73</td>
</tr>
<tr>
<td></td>
<td>( \Delta% )</td>
<td>-0.173</td>
</tr>
<tr>
<td>III</td>
<td>Theory</td>
<td>1.29</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>1.25</td>
</tr>
<tr>
<td></td>
<td>( \Delta% )</td>
<td>3.18</td>
</tr>
<tr>
<td>IV</td>
<td>Theory</td>
<td>4.37</td>
</tr>
<tr>
<td></td>
<td>FEM</td>
<td>4.29</td>
</tr>
<tr>
<td></td>
<td>( \Delta% )</td>
<td>1.90</td>
</tr>
</tbody>
</table>

\( \Delta\% = 100(\text{U}^*_{\text{theory}} - \text{U}^*_{\text{FEM}}) / \text{U}^*_{\text{theory}} \)

References

Proceedings of the

Second International Conference of the European Society for Wood Mechanics

May 25th – 28th, 2003, Stockholm, Sweden

Lennart Salmén
Editor