EFFECTS OF SHEAR ON DEFLECTION OF WOOD CANTILEVER LOADED AT FREE END

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Deflection functions of a wood cantilever having a rectangular cross section are presented. Wood is considered an orthotropic material with distinct mechanical properties in three mutually perpendicular directions. We consider a cantilever in the radial and longitudinal plane of wood.

General Anisotropic Elasticity

Let axes 1 and 2 define the principal material plane with axis 1 in the grain direction and axis 2 in the radial direction. The geometrical axes x and y are located at the free end of the beam with axis x at an angle q from the 1 axis (Fig. 1). The stress/strain relations in anisotropic elasticity theory are [1]:

$$\varepsilon_x = S_{xx}\sigma_x + S_{xy}\sigma_y + S_{xs}\tau_{xy} \tag{1a}$$

$$\varepsilon_y = S_{yx}\sigma_x + S_{yy}\sigma_y + S_{ys}\tau_{xy} \tag{1b}$$

$$\gamma_{xy} = S_{sx}\sigma_x + S_{sy}\sigma_y + S_{ss}\tau_{xy} \tag{1c}$$

where the transformed compliances S_{ij} can be given as

$$\begin{aligned}
S_{xx} &= m^4S_{11} + n^4S_{22} + 2m^2n^2S_{12} + m^2n^2S_{66}\\
S_{yy} &= n^4S_{11} + m^4S_{22} + 2m^2n^2S_{12} + m^2n^2S_{66}\\
S_{xy} &= S_{yx} = m^2n^2S_{11} + m^2n^2S_{22}\\
&\quad + (m^4+n^4)S_{12} - m^2n^2S_{66}\\
S_{xs} &= S_{sx} = 2m^3nS_{11} - 2mn^3S_{22}\\
&\quad +2(mn^3 - m^3n)S_{12} + (mn^3 - m^3n)S_{66}\\
S_{ys} &= S_{sy} = 2mn^3S_{11} - 2m^3nS_{22}\\
&\quad +2(m^3n - mn^3)S_{12} + (m^3n - mn^3)S_{66}\\
S_{ss} &= 4m^2n^2S_{11} + 4m^2n^2S_{22}\\
&\quad -8m^2n^2S_{12} + (m^2-n^2)^2S_{66}
\end{aligned} \tag{2}$$

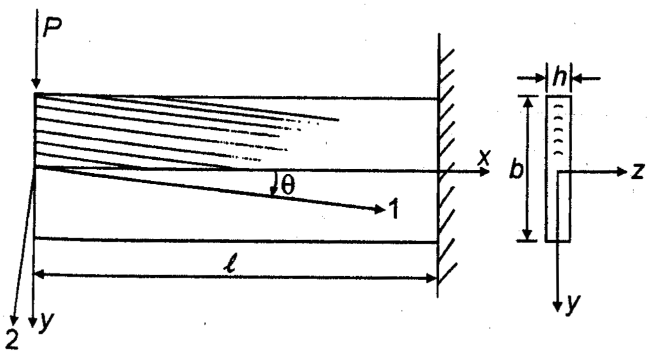

Fig. 1—An orthotropic cantilever subjected to a single load (x and y are geometrical axes; 1 and 2 are material axes).

with $m = \cos$ q and $n = \sin$ q and the principal compliances

$$\begin{aligned}
S_{11} &= \frac{1}{E_1}, \quad S_{22} = \frac{1}{E_2},\\
S_{12} &= -\frac{\upsilon_{12}}{E_1} = S_{21} = -\frac{\upsilon_{21}}{E_2}, \quad S_{66} = \frac{1}{G_{12}}
\end{aligned} \tag{3}$$

In addition, the strain/displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x} \tag{4a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \tag{4b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{4c}$$

Derivation of Deflection Functions

Stress components of a cantilever subjected to a single load P at the free end are [2]

$$\begin{aligned}
\sigma_x &= -\frac{P}{I}xy + \frac{PS_{xs}}{IS_{xx}}\left(\frac{b^2}{12} - y^2\right)\\
\sigma_y &= 0\\
\tau_{xy} &= -\frac{P}{2I}\left(\frac{b^2}{4} - y^2\right)
\end{aligned} \tag{5}$$

where $I = \dfrac{hb^3}{12}$.

From Equations (1a), (5), and (4a), we obtain

$$u = -\frac{P}{2I}\left[S_{xx}x^2y + \frac{S_{xs}}{12}(b^2 + 12y^2)x\right] + f(y) \tag{6}$$

From Equations (1b), (5), and (4b), it follows

$$\begin{aligned}
v = \frac{P}{2I}\Bigg[&\frac{b^2}{12S_{xx}}(2S_{xy}S_{xs} - 3S_{xx}S_{ys})y\\
&- S_{xy}xy^2 + \frac{(S_{xx}S_{ys} - 2S_{xy}S_{xs})}{3S_{xx}}y^3\Bigg] + g(x)
\end{aligned}$$

The functions $f(y)$ and $g(x)$ can be obtained from Equations (1c) and (4c), the boundary conditions, and a constraining condition, eliminating the possibility of rotation of the fixed end in the xy plane [3]. Two possible constraining conditions are considered: (1) When an element of the axis of the beam is fixed at the fixed end, we have

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=l\\y=0}} = 0 \tag{8}$$

which leads to the deflection function

$$v = \frac{P}{2I}\left[\frac{b^2}{12S_{xx}}(2S_{xy}S_{xs} - 3S_{xx}S_{ys})y - S_{xy}xy^2 + \frac{(S_{xx}S_{ys} - 2S_{xy}S_{xs})}{3S_{xx}}y^3\right] + \frac{PS_{xx}}{2I}\left[\frac{x^3}{3} - (x-l)l^2 - \frac{l^3}{3}\right] \tag{9}$$

and at the free end

$$v_{\substack{x=0\\y=0}} = \frac{PS_{xx}l^3}{3I} \tag{10}$$

(2) When a vertical element at the fixed end is fixed, we have

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} = 0 \tag{11}$$

The corresponding deflection function becomes

$$\begin{aligned} v = {} & \frac{P}{2I}\left[\frac{b^2}{12S_{xx}}(2S_{xy}S_{xs} - 3S_{xx}S_{ys})y - S_{xy}xy^2 + \frac{(S_{xx}S_{ys} - 2S_{xy}S_{xs})}{3S_{xx}}y^3\right] \\ & + \frac{PS_{xx}}{6I}x^3 + \frac{P}{2I}\left[\frac{b^2}{2}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right) - S_{xx}l^2\right]x \\ & + \frac{Pl}{I}\left[\frac{S_{xx}l^2}{3} - \frac{b^2}{4}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right)\right] \end{aligned} \tag{12}$$

At the free end

$$v_{\substack{x=0\\y=0}} = \frac{PS_{xx}l^3}{3I} - \frac{Pb^2l}{4I}\left(\frac{S_{xs}^2}{3S_{xx}} - \frac{S_{ss}}{2}\right) \tag{13}$$

which is the corrected form of the corresponding equation in [4]. Note the first term is identical to Equation (10).

Shear Effects on Deflection

For isotropic material, the first term in Equation (13) is due to flexual and the second term due to shear [3]. But for orthotropic material, that is only true when q = 0. When q ± 0, S_{xx} in the first term as well as S_{xs} and S_{ss} are all functions of the principal compliances as shown in Equations (2) and (3). It is therefore of interest, after the transformed compliances are replaced by the principal compliances, to separate the terms containing S_{66} from those that do not in Equation (13). For this separation the only term that requires special attention is the ratio S_{xs}^2/S_{xx}. For abbreviation, let S_{xs} = a + ßS_{66} and S_{xx} = c + dS_{66} (see Equation (2)). Then we have

$$\frac{S_{xs}^2}{S_{xx}} = \frac{(\alpha+\beta S_{66})^2}{\chi+\delta S_{66}} = \frac{\alpha^2[1+(\beta/\alpha)S_{66}]^2}{\chi[1+(\delta/\chi)S_{66}]} = \frac{\alpha^2}{\chi}\frac{(1+X)}{(1+Y)} = \frac{\alpha^2}{\chi}\left[1+(X-Y)(1-Y+Y^2-Y^3+\ldots.)\right] \tag{14}$$

For $|Y|<1$ as in the present case, the series in Equation (14) converges very fast. All the terms containing X and Y are functions of S_{66}, although they are not devoid of the other principal compliances. The separation mentioned before is now straightforward.

Results and Conclusions

Mechanical properties for Sitka spruce [5] are used for numerical calculations: E_1 = 1 1,800 MPa, E_2 = 2,216 MPa, G_{12} = 910 MPa, u_{12} = 0.37. The geometrical dimensions and the applied load in Fig. 1 are l = 80 mm, b = 40 mm, h = 10 mm and P = 100 N. For the grain slope, the assumed values are 0°, 5°, 10°, and 15°. Results are shown in Table 1. The transformed compliances referred to the geometrical axes cannot show the effects of shear on deflection the same as do the principal compliances referred to the material axes. As the grain slope increases, deflection increases with shear (terms containing S_{66}) only.

Table 1. Deflection of Sitka spruce cantilever [Eq. (13)]

q,Deg	Flexural[a] (mm)	Shear[a] (mm)	Flexural[b] (mm)	Shear[b] (mm)
0	2.71e-2	3.30e-2	2.71e-2	3.30e-2
5	2.92e-2	3.13e-2	2.70e-2	3.35e-2
10	3.53e-2	2.75e-2	2.69e-2	3.60e-2
15	4.50e-2	2.35e-2	2.71e-2	4.14e-2

[a] Terms without S_{xs} and S_{ss} under Flexual; others under Shear.
[b] Terms without S_{66} under Flexual; others under Shear.

References

1. Tsai, S.W. and Hahn, H.T. 1980. Introduction to composite materials. Technomic Publishing Co., Inc., Lancaster, PA.
2. Lekhnitskii, S.G. 1968. Anisotropic plates. Gordon and Breach Science, New York.
3. Timoshenko, S. and Goodier, J.N. 1951. Theory of elasticity. McGraw–Hill, New York.
4. Kilic, O., Aktas, A. and Dirikolu, M.H. 2001. An investigation of the effects of shear on the deflection of an orthotropic cantilever beam by the use of anisotropic elasticity theory. Composites Science and Technology, 61(14):2055-2061.
5. Liu, J.Y. 2000. Effects of shear coupling on shear properties of wood. Wood and Fiber Science, 32(4): 458-465.

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