DEFLECTIONS AND STRESSES IN
CIRCULAR TAPERED BEAMS AND POLES

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ABSTRACT
Equations are derived for calculating the deflections and stresses in circular tapered beams and poles. Cantilevers and simple beams are analyzed for concentrated, uniform, linearly increasing, trapezoidal, parabolic, and self-weight loads.

A set of graphs provides a rapid means for determining the locations and magnitudes of maximum deflections and stresses.

Introduction
Wood logs are used in their natural round form in many structural applications. As simple beams, they serve as the main structural elements of bridges, especially in remote areas, and as roof beams in log cabins and similar rustic types of construction. As cantilevers, logs are used extensively for transmission structures and as vertical members in pole-frame construction.

The stresses and deflections of tapered logs are usually computed by analyzing an "equivalent" prismatic beam. However, the greater the taper, the less accurate are such approximations. No compilation of equations for calculating strength and stiffness of round tapered beams appears in the literature.

In this paper, equations are derived for calculating the deflections and stresses in tapered wood poles. Both cantilevers and simple beams are analyzed under several different types of load.

Scope and Limitations
It is assumed that the cross section of the pole is perfectly circular and that the taper is uniform (i.e., linear) from tip to butt. Only flexure is considered; it is assumed that the beams are sufficiently slender so that shear...
deformations are negligible and that no axial loads exist. The modulus of elasticity is assumed to be constant.

Only the major steps in the derivations are presented. The complete derivations, showing all the intermediate steps, are quite lengthy.

Symbol

\( L \) Length of the pole,
\( d_A \) Tip diameter,
\( d_B \) Butt diameter,
\( d \) Diameter at any point, \( d = d_A[1 + (r - 1)g] \)
\( r \) Ratio of butt diameter to tip diameter, \( \frac{1}{r} = \frac{d_B}{d_A} \),
\( X \) Distance from the tip,
\( \xi \) Normalized distance, \( \xi = \frac{x}{L} \),
\( \alpha \) Normalized location of a concentrated load,
\( I_A \) Moment of inertia at the tip, \( I_A = \frac{\pi d_A^4}{64} \),
\( I \) Moment of inertia at any point, \( I = I_A \left[1 + (r - 1)\xi\right]^4 \),
\( S_A \) Section modulus at the tip, \( S_A = \frac{\pi}{32} d_A^3 \),
\( S \) Section modulus at any point, \( S = S_A \left[1 + (r - 1)\xi\right]^3 \),
\( E \) Modulus of elasticity (assumed constant),
\( Y \) Deflection of the beam,

\[ Y \]

[It is assumed that \( r \geq 1 \). However, some of the equations presented are also applicable to the case of \( r < 1 \).]
M  Bending moment,

f  Bending stress.

The following subscripts are also used:

A,B  denote values at the tip and butt, respectively,
L,R,P  denote values to the left of, to the right of, and directly under a concentrated load.
max  denotes a maximum value.

Cantilevers

For a cantilever beam loaded in one direction, the maximum deflection is always at the tip. To solve for this deflection, a fictitious unit load is placed at the tip, and the deflection can be calculated by the method of virtual work as:

\[ y_A = \int_0^L \frac{M_m}{EI} \, dx \]  

(1)

where \( y_A \) is the tip deflection,

M  is the bending moment due to the real load,

m  is the bending moment due to the fictitious load.

Noting that \( m = x = \xi L \) and \( I = I_A [1 + (r - 1)\xi]^4 \), this can be rewritten as:

\[ y_A = \frac{L^2}{EI_A} \int_0^1 \frac{M_m}{\left[1 + (r - 1)\xi\right]^4} \, d\xi \]  

(2)

This equation will be utilized in the following analyses of cantilevers subjected to various types of loadings.

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Concentrated Tip Load

For a concentrated load \( P \) at the tip, the bending moment is:

\[
M = P \ell_x = \frac{PL}{3EIA}
\]

(3)

and the tip deflection (Eq. 2) is therefore:

\[
y_A = \frac{PL^3}{3EIA} \int_0^1 \frac{x^2}{1 + (r - 1)x} \, dx
\]

(4)

This simplifies to:

\[
y_A = \frac{PL^3}{3EIA} \cdot \frac{1}{r^3}
\]

(5)

In the above equation, the first factor \( (PL^3 / 3EIA) \) represents the deflection of a prismatic beam (i.e., no taper, \( r = 1 \)); the second factor \( (1/r^3) \) is the modification required to account for taper. This same partitioning will be used in all subsequent equations for tip deflection and for maximum bending stress in a cantilever.

The bending stress in the beam, \( f \), is

\[
f = \frac{M}{S_A} = \frac{PL\xi}{S_A [1 + (r - 1)\xi]^3}
\]

(6)

The location of maximum stress is determined by:

\[
\frac{df}{d\xi} = 0
\]

(7)

which yields:

\[
\xi = \frac{1}{2(r - 1)} \quad \text{for} \quad r > 1.5
\]

(8)

The corresponding maximum stress, obtained by substituting Eq. 8 into Eq. 6, and the pole diameter at the point of maximum stress are:

\[
f_{max} = \frac{PL}{S_A} \cdot \frac{4}{27(r - 1)} \quad \text{for} \quad r > 1.5
\]

(9)

\[
d = 1.5 \cdot d_A
\]

(10)
For $r < 1.5$, the maximum stress is at the support (butt) and is equal to:

$$f_{\text{max}} = \frac{PL}{SA} \cdot \frac{1}{r^3} \quad \text{for } r < 1.5$$

(11)

Curves which define the normalized tip deflection (i.e., the second factor in Eq. 5), the normalized maximum stress (the second factors in Eqs. 9 and 11), and the location of maximum stress (Eq. 8) are presented in Figure 1A.

**Uniform Load**

For a uniform load $w$, the bending moment is

$$M = \frac{1}{2} wL^2 \xi^2$$

(12)

which gives for the tip deflection (Eq. 2),

$$y_A = \frac{wL^4}{2EI_A} \int_0^1 \frac{\xi^3}{\left[1 + (r - 1)\xi\right]^4} \, d\xi$$

(13)

Carrying out the integration and simplifying yields:\n
$$y_A = \frac{wL^4}{8EI_A} \cdot \frac{4}{(r - 1)^3} \left[\ln \frac{r}{r - 1} - \frac{(11r^2 - 7r + 2)}{6r^3}\right]$$

(14)

Following the same procedure as before (Eqs. 7–9), the location and magnitude of maximum stress and the corresponding diameter are:

$$\xi = \frac{2}{r - 1}$$

(15)

$$f_{\text{max}} = \frac{wL^2}{2SA_d} \cdot \frac{4}{27(r - 1)^2} \quad \text{for } r > 3$$

(16)

$$d = 3d_A$$

(17)

For $r < 3$, the maximum stress at the support is

3/ Eq. 14 and many subsequent equations contain the factor $(r - 1)$ in the denominator. Thus, they cannot be applied directly to the case of $r = 1$. The reader is also cautioned that when using hand-held calculators, round-off error may also produce erroneous results when $r$ is close to unity.
Figure 1B presents a plot of normalized deflection (second factor, Eq. 14) and stress (second factor, Eqs. 16 and 18) and location of maximum stress (Eq. 15).

**Increasing Load**

For a distributed load which increases linearly from zero at the tip to \( w \) at the butt (i.e., a "ramp" load), the bending moment is:

\[
M = \frac{1}{6} w L^2 \xi^3
\]  

and Eq. 2 becomes

\[
y_A = \frac{wL^4}{6EI_A} \int_0^1 \frac{\xi^4}{[1 + (r - 1)\xi]^4} d\xi
\]

Performing the indicated integration yields

\[
y_A = \frac{wL^4}{30EI_A} \cdot \frac{5}{(r - 1)^4} \left[\frac{(3r^3 + 13r^2 - 5r + 1)}{3r^2} \frac{-4 \ln r}{(r - 1)}\right]
\]

The maximum bending stress for this loading is always at the support and is equal to

\[
f_{\text{max}} = \frac{wL^2}{6S_A} \cdot \frac{1}{r^3}
\]

Normalized deflection and stress (the second factors in Eqs. 21 and 22, respectively) are plotted in Figure 1C.

**Trapezoidal Load**
A trapezoidal load that increases linearly from \( w \) at the tip to \( pw \) at the support can be viewed as the sum of two loads: a uniform load of magnitude \( w \) and a ramp load of \((\rho - 1)w\). Therefore, the bending moment can be computed as the sum of Eq. 12 (for the uniform load) plus \((\rho - 1)\) times Eq. 19 (for the ramp load),

\[
M = \frac{1}{6} wL^2 \left[ 3\xi^2 + (\rho - 1)\xi^3 \right] \tag{23}
\]

and the deflection can be computed by adding Eq. 14 to \((\rho - 1)\) times Eq. 21.

The bending stress is

\[
f = \frac{M}{S} = \frac{wL^2}{6S_A} \left[ 3\xi^2 + (\rho - 1)\xi^3 \right] \tag{24}
\]

Again, setting \( \frac{df}{d\xi} = 0 \) locates the point of maximum stress

\[
f_{\text{max}} = \frac{wL^2(\rho + 2)}{6S_A} \cdot \frac{4}{(\rho + 2)(3r - \rho - 2)^2} \tag{25}
\]

and the corresponding stress and diameter are:

\[
d = d_A \left[ 1 + 2\frac{(r - 1)}{(r - \rho)} \right] \tag{26}
\]

For \((r - p) < 2\), the maximum stress is at the support and equal to:

\[
f_{\text{max}} = \frac{wL^2(\rho + 2)}{6S_A} \cdot \frac{1}{r^3} \tag{27}
\]

For the special case of \( \rho = r \), which occurs when poles are subjected to wind loading, the tip deflection (Eq. 14 plus \((r - 1)\) times Eq. 21) is:

\[
y_A = \frac{wL^4}{8EI} \cdot \frac{4}{3(r - 1)^3} \left[ \frac{(6r^3 - 7r^2 + 11r - 4)}{6r^3} - \frac{1n r}{(r - 1)} \right] \tag{28}
\]

and the maximum stress (Eq. 18 plus \((r - 1)\) times Eq. 22), which is always at the support, is

\[
f_{\text{max}} = \frac{wL^2}{2S_A} \cdot \frac{(r + 2)}{3r^3} \tag{29}
\]

The second factors of Eqs. 29 and 30 define the normalized deflection and stress for a trapezoidal load with \( p = r \). They are plotted in Figure 1D.
Parabolic Load

For a load which increases parabolically from zero at the tip to \( w \) at the support, the bending moment is:

\[
M = \frac{1}{12} w L^2 \xi^4
\]  

(31)

and the deflection, Eq. 2, is

\[
y_A = \frac{wL^4}{12EI_A} \int_0^1 \frac{\xi^5}{\left[1 + (r - 1)\xi^4\right]^4} \, d\xi
\]

(32)

Evaluating the integral gives

\[
y_A = \frac{wL^4}{72EI_A} \cdot \frac{1}{(r - 1)^5} \left[ \frac{(3r^4 - 27r^3 - 47r^2 + 13r - 2)}{r^3} + \frac{60 \ln r}{(r - 1)} \right]
\]

(33)

The normalized deflection and stress, from Eqs. 33 and 34, are plotted in Figure 1E.

Self-Weight Load

The load on a beam due to its own weight can be considered as the sum of three loads: a uniform load \( w \), a ramp load of \( 2(r - 1)w \), and a parabolic load of \( (r - 1)^2w \). Thus, the tip deflection of a cantilever loaded by its own weight is computed from Eq. 14 plus \( 2(r - 1) \) times Eq. 21 plus \( (r - 1)^2 \) times Eq. 33:
The stress in the beam is:

\[ f = \frac{M}{S} = \frac{wL^2}{128A} \left[ 6 \xi^2 + 4(r - 1) \xi^3 + (r - 1)^2 \xi^4 \right] \left[ 1 + (r - 1) \xi \right]^3 \]  

Setting \( \frac{df}{d\xi} = 0 \) yields no solution in the range \( 0 \leq \xi \leq 1 \). Therefore, the maximum stress is always at the support \( (\xi = 1) \) and equal to:

\[ f_{\text{max}} = \frac{wL^2}{28A} \cdot \frac{(r^2 + 2r + 3)}{6r^3} \]  

The normalized tip deflection and maximum stress, from Eqs. 35 and 37, are plotted in Figure IF.

**Simple Beams**

Unlike cantilevers, the point of maximum deflection is not known for simply supported tapered beams. Therefore, a different technique is employed in analyzing this type of beam.

Curvature, the second derivation of deflection, is related to bending moment by

\[ \frac{d^2 y}{dx^2} = -\frac{M}{EI} \]  

Integrating once gives the slope, \( \frac{dy}{dx} \), and integrating a second time yields the deflection, \( y \). The constants of integration are evaluated by considering boundary conditions and continuity.

In the following derivations, the following notation is used for simplicity and brevity:

\[ K \equiv 1 + (r - 1) \xi \]

\[ K_\alpha \equiv 1 + (r - 1) \alpha \]

**Stationary Point Load**
For a simple beam subjected to a point load $P$ at a distance $\xi_L$ from the left support, the bending moment to the left of the load is

$$M_L = PL(1 - \alpha)\xi$$  \hfill (39)

Then, from Eq. 38,

$$\frac{d^2y_L}{dx^2} = -\frac{M_L}{EI} = \frac{PL}{EI_A} (\alpha - 1) \frac{\xi}{K^4}$$  \hfill (40)

Integrating once gives the slope

$$\frac{dy_L}{dx} = \frac{PL^2(\alpha - 1)}{EI_A(r - 1)^2} \left[ -\frac{1}{2K^2} + \frac{1}{3K} + A \right]$$  \hfill (41)

and integrating a second time gives the deflection

$$y_L = \frac{PL^3(\alpha - 1)}{EI_A(r - 1)^3} \left[ \frac{1}{2K - 1} - \frac{1}{6K^2} + (r - 1)A\xi + B \right]$$  \hfill (42)

where $A$ and $B$ are integration constants.

To the right of the load, the bending moment is

$$M_R = PL\alpha(1 - \xi)$$  \hfill (43)

and the curvature is

$$\frac{d^2y_R}{dx^2} = -\frac{M_R}{EI} = \frac{PL}{EI_A} \frac{\alpha (\xi - 1)}{K^4}$$  \hfill (44)

Again, integrating twice gives the slope and deflection:

$$\frac{dy_R}{dx} = \frac{PL^2\alpha}{EI_A(r - 1)^2} \left[ -\frac{1}{2K^2} + \frac{r}{3K^3} + C \right]$$  \hfill (45)

$$y_R = \frac{PL^3\alpha}{EI_A(r - 1)^3} \left[ \frac{1}{2K} - \frac{r}{6K^2} + (r - 1)C(\xi - 1) + D \right]$$  \hfill (46)

The integration constants $A$, $B$, $C$, and $D$ are evaluated by considering the boundary and continuity conditions:

- at $\xi = 0$, $y_L = 0$
- at $\xi = 1$, $y_R = 0$
- at $\xi = \alpha$, $y_L = y_R$ and $\frac{dy_L}{dx} = \frac{dy_R}{dx}$
Solving the resulting equations simultaneously gives:

\[
A = \frac{1}{6K_\alpha} + \frac{\alpha(r - 1)}{3rK_\alpha}
\]
\[
B = -\frac{1}{3}
\]
\[
C = \frac{1}{6K_\alpha} - \frac{(1 - \alpha)(r - 1)}{3rK_\alpha}
\]
\[
D = -\frac{1}{3r}
\]

(47)

Substituting Eqs. 47 back into Eqs. 42 and 46 and simplifying gives:

\[
y_L = \frac{PL^3(1 - \alpha)\xi}{6EI_A r K_\alpha^2 K_a^2} \left[ 2\alpha(1 - \xi)K_aK - r(\alpha - \xi)^2 \right]
\]

(48)

\[
y_R = \frac{PL^3\alpha(1 - \xi)}{6EI_A r K_\alpha^2 K_a^2} \left[ 2(1 - \alpha)K_aK - r(\alpha - \xi)^2 \right]
\]

(49)

The deflection directly under the load, found by setting \( \xi = \alpha \) in either Eq. 48 or 49, is

\[
y_p = \frac{PL^3\alpha^2 (1 - \alpha)^2}{3EI_A r [1 + (r - 1)\alpha]^2}
\]

(50)

The maximum deflection can be found by determining the point of zero slope from either Eq. 41 or Eq. 45 and substituting the obtained value of \( \xi \) into Eq. 48 or Eq. 49.

The maximum stress is always to the left of, or directly under, the load. The bending stress in the left segment of the beam is

\[
f = \frac{M_L}{S} = \frac{PL(1 - \alpha)\xi}{S_A [1 + (r - 1)\xi]^3}
\]

(51)

which is a maximum at

\[
\xi = \frac{1}{2(r - 1)} \quad \text{for} \quad \alpha > \frac{1}{2(r - 1)}
\]

(52)

\[
d = 1.5d_A
\]

(53)
The corresponding values of maximum stress are

\[ f_{\text{max}} = \frac{4PL(1 - \alpha)}{27SA(r - 1)} \quad \text{for } \alpha > \frac{1}{2(r - 1)} \]  
\[ f_{\text{max}} = \frac{PL\alpha(1 - \alpha)}{S_A (1 + (r - 1)\alpha)^3} \quad \text{for } \alpha < \frac{1}{2(r - 1)} \]  

For the special case of a midspan load \((\alpha = 1/2)\):

\[ y_L = \frac{PL^3}{48EI_A} \cdot \frac{4\xi}{r(r + 1)^2K^2} \left[ 2K(r + 1)(1 - \xi) - r(1 - 2\xi)^2 \right] \]  
\[ f_{\text{max}} = \frac{PL}{4S_A} \cdot \frac{8}{27(r - 1)} \quad \text{for } r > 2 \]  
\[ f_{\text{max}} = \frac{PL}{4S_A} \cdot \frac{8}{(r + 1)^3} \quad \text{for } r < 2 \]  

The magnitudes of normalized maximum deflection and stress and their locations are plotted in Figure 2G.

**Moving Point Load**

The previous section analyzes a tapered beam subjected to a concentrated load at a prescribed location \(a\). For a moving load, there is a certain location for which the deflection (directly under the load) is an absolute maximum and another location for which the bending stress (also directly under the load) is maximized.

The location for maximum deflection is found by differentiating Eq. 50, the deflection beneath the load, and setting to zero. This gives:

\[ \alpha = \sqrt{\frac{r - 1}{r - 1}} \]  
\[ d = \sqrt{\frac{r}{r - 1}} \]  

The corresponding (absolute) maximum deflection is

\[ y_{\text{max}} = \frac{PL^3}{48EI_A} \cdot \frac{16}{r} \left[ \frac{\sqrt{r - 1}}{r - 1} \right]^4 \]
The load location for maximum stress (Eq. 57) is similarly determined to be

$$\alpha = \frac{r - \sqrt{r^2 - r + 1}}{r - 1}$$

(64)

and the stress itself can be computed by substituting Eq. 64 into Eq. 57:

$$f_{\text{max}} = \frac{PL}{4S_A} \cdot \frac{4\alpha(1 - \alpha)}{K_\alpha^3}$$

(65)

The locations of load which produce maximum deflection and stress (Eqs. 61 and 64, respectively) and the corresponding normalized deflection and stress (from Eqs. 63 and 65) are plotted in Figure 2H.

**Uniform Load**

For a uniform load, w, the bending moment in a simple beam is

$$M = \frac{1}{2} wL^2 \xi (1 - \xi)$$

(66)

and the curvature is

$$\frac{d^2y}{dx^2} = -\frac{M}{EI} = \frac{wL^2 \xi (\xi - 1)}{2EI_A K^4}$$

(67)

Integration gives the slope and deflection:

$$\frac{dy}{dx} = \frac{wL^3}{2EI_A (r - 1)^3} \left[ -\frac{1}{K} + \frac{(r + 1)}{2K^2} - \frac{r}{3K^3} + \frac{A}{(r - 1)} \right]$$

(68)

$$y = \frac{wL^4}{2EI_A (r - 1)^4} \left[ -\ln K - \frac{(r + 1)}{2K} + \frac{r}{6K^2} + \frac{A\xi + B}{(r - 1)} \right]$$

(69)

From the boundary conditions of zero deflection at the supports (y = 0 at $\xi = 0$ and $\xi = 1$), the integration constants are determined:

$$A = \ln r - \frac{(r^2 - 1)}{3r}$$

(70)
The bending stress is

\[ f = \frac{M}{S} = \frac{wL^2 \xi(1 - \xi)}{2S_A[1 + (r-1)\xi]}^3 = \frac{wL^2}{8S_A} \cdot \frac{4\xi(1 - \xi)}{K^3} \] (72)

and is a maximum at

\[ \xi = \frac{r - \sqrt{r^2 - r + 1}}{r - 1} \] (73)

The locations of maximum deflection (i.e., zero slope, Eq. 68) and maximum stress (Eq. 73) and the magnitudes of the corresponding normalized deflection and stress (Eqs. 69 and 72) are plotted in Figure 21.

**Increasing Load**

For a load which increases linearly from zero at the tip to \( w \) at the butt, the bending moment is

\[ M = \frac{1}{6} wL^2 \xi(1 - \xi^2) \] (74)

which gives for curvature, slope, and deflection:

\[ \frac{d^2y}{dx^2} = \frac{M}{EI} = \frac{wL^2}{6EI_A} \frac{\xi(\xi^2 - 1)}{K^4} \] (75)

\[ \frac{dy}{dx} = \frac{wL^3}{6EI_A(r - 1)^4} \left[ \ln K + \frac{3}{K} + \frac{(r^2 - 2r - 2)}{2K^2} + \frac{(-r^2 + 2r)}{3K^3} + \frac{A}{(r - 1)} \right] \] (76)

\[ y = \frac{wL^4}{6EI_A(r - 1)^5} \left[ K(\ln K - 1) + 3 \ln K - \frac{(r^2 - 2r - 2)}{2K} + \frac{(r^2 - 2r)}{6K^2} + A\xi + B \right] \]

\[ = \frac{0.06522 \cdot wL^4}{EI_A} \cdot \frac{25.55}{(r - 1)^5} \left[ \right] \] (77)
where

\[
A = -\left[\frac{(r - 1)(2r^2 - 11r - 4)}{6r} + (r + 3) \ln r\right]
\]

\[
B = \frac{1}{3} r(r - 2)
\]

The bending stress is equal to

\[
f = \frac{M}{S} = \frac{\omega L^2 \xi (1 - \xi^2)}{6 S_A [1 + (r - 1) \xi]^3} = \frac{0.06415 \omega L^2}{S_A} \cdot \frac{2.598 \xi (1 - \xi^2)}{K^3}
\]

which is a maximum at

\[
\xi = 1 - r + \frac{\sqrt{r^2 - 2r + 4}}{3}
\]

The locations of the maximum deflection (zero slope, Eq. 76) and maximum stress (Eq. 81) are plotted in Figure 2J along with the corresponding values of deflection and stress (Eqs. 77 and 80).

**Trapezoidal Load**

Again considering a trapezoidal load to be the sum of uniform load \(w\) and ramp load \((p - 1)w\), the bending moment (from Eqs. 66 and 74) is equal to

\[
M = \frac{1}{6} \omega L^2 \xi \left[3(1 - \xi) + (\rho - 1)(1 - \xi^2)\right]
\]

and the slope can be computed from Eqs. 68 and 76 and the deflection from Eqs. 69 and 77.

The bending stress is

\[
f = \frac{M}{S} = \frac{\omega L^2 \xi}{6 S_A} \left[\frac{3(1 - \xi) + (\rho - 1)(1 - \xi^2)}{[1 + (r - 1) \xi]^3}\right]
\]

which is a maximum when

\[
3(r - \rho) \xi^2 - 2 \left[(r - 1)(\rho + 2) + 3\right] \xi + (\rho + 2) = 0
\]

For the special case of \(\rho = r\), which occurs when a beam is subjected to snow load or similar distributed loading, Eq. 84 gives for the location of maximum stress:
and the stress (Eq. 83) becomes:

\[
\xi = \frac{(r + 2)}{2(r^2 + r + 1)}
\]

\[\text{(85)}\]

The deflection is:

\[
\frac{f}{8S_A} = \frac{wL^2}{8S_A} \cdot \frac{4\xi(1 - \xi)[3 + (r - 1)(1 + \xi)]}{3K^3}
\]

\[\text{(86)}\]

The locations of maximum stress (Eq. 85) and maximum deflection (zero slope) and their normalized magnitudes (Eqs. 86 and 87) are plotted in Figure 3K for this special case, \(\varphi = r\).

**Parabolic Load**

For a simple beam under a load that increases parabolically from zero at the tip to \(w\) at the butt, the bending moment is

\[
M = \frac{1}{12}wL^2\xi(1 - \xi^3)
\]

\[\text{(88)}\]

and the curvature, slope, and deflection are

\[
\frac{d^2y}{dx^2} = -\frac{M}{EI} = \frac{wL^2}{12EI_A} \frac{\xi(\xi^3 - 1)}{K}
\]

\[\text{(89)}\]

\[
\frac{dy}{dx} = \frac{wL^3}{12EI_A(r - 1)^5} \left[ K - 4 \ln K - \frac{6}{K} + \frac{(r^3 - 3r^2 + 3r + 3)}{2K^2} \right.
\]

\[
- \frac{r^3 - 3r^2 + 3r}{3K^3} + \frac{A}{(r - 1)} \]
The bending stress is equal to

\[ \frac{wL^4}{12EI_A} \left[ \frac{K^2}{2} - 4K(\ln K - 1) - 6\ln K - (r^3 - 3r^2 + 3r + 3) \right] \]

\[ + \frac{(r^3 - 3r^2 + 3r)}{6K^2} + A + B \]

\[ = 0.03883 \frac{wL^4}{EI_A} \cdot 21.46 \left[ \frac{K^2}{2} - 4K(\ln K - 1) - 6\ln K - (r^3 - 3r^2 + 3r + 3) \right] \]

where

\[ A = 2(2r + 3)\ln r - \frac{(r - 1)(r^3 - 2r^2 + 18r + 3)}{3r} \]

\[ B = \frac{(r - 1)^3 - 8}{3} \]

The locations of maximum deflection (zero slope, Eq. 89) and maximum stress (Eq. 95) and their normalized magnitudes (Eqs. 91 and 94) are plotted in Figure 3L.

**Self-weight Load**

Again taking self-weight load to be the sum of uniform load \( w \), ramp load \( 2(r - 1)w \), and parabolic load \( (r - 1)^2 w \), the deflection (from Eqs. 69, 77, and 91) can be written as:

\[ y = \frac{5wL^4}{384EI_A} \cdot \frac{16}{15} \left[ \frac{r + 2 + (2r^2 + 3r - 2)\xi - 3r\xi^2}{K^2r} \right] \]

and the bending stresss (Eqs. 72, 80, and 94) as:

\[ f = \frac{wL^2}{8S_A} \cdot \frac{2\xi(1 - \xi)}{3K^3} \left[ 6 + 4(r - 1)(1 + \xi) + (r - 1)^2(1 + \xi + \xi^2) \right] \]
The locations of maximum deflection and stress and their normalized magnitudes (Eqs. 96 and 97) are plotted in Figure 3M.

**Prismatic Beams**

Equations for the deflection of prismatic beams (i.e., no taper, \( r = 1 \)) are presented in this section.

For a point load, the deflection of a prismatic beam to the left of, to the right of, and directly under the load (cf. Eqs. 48-50 for tapered beam) are

\[
y_L = \frac{PL^3}{6EI} (1 - \alpha) \xi \left[ 2\alpha - \alpha^2 - \xi^2 \right]
\]

(98)

\[
y_R = \frac{PL^3}{6EI} \alpha (1 - \xi) \left[ 2\xi - \alpha^2 - \xi^2 \right]
\]

(99)

\[
y_P = \frac{PL^3}{3EI} \alpha^2 (1 - \alpha)^2
\]

(100)

For a uniform load \( w \) (cf. Eq. 69):

\[
y = \frac{wL^4}{24EI} \xi (\xi^3 - 2\xi^2 + 1)
\]

(101)

For a linearly increasing load (cf. Eq. 77):

\[
y = \frac{wL^4}{360EI} \xi (3\xi^4 - 10\xi^2 + 7)
\]

(102)

For a parabolic load (cf. Eq. 91):

\[
y = \frac{wL^4}{360EI} \xi (\xi^5 - 5\xi^2 + 4)
\]

(103)

**Conclusion**

The preceding sections have presented equations for determining deflections and stresses in round poles and beams. Their locations, as well as their magnitudes, are dependent upon the taper. The equations were derived for circular sections, but, except for the self-weight loads, can also be adapted to any member in which the moment of inertia, \( I \), changes with the fourth power of distance and in which the section modulus, \( S \), changes with the cube of distance along the beam.

The figures provide a fast and simple means for determining the locations and magnitudes of maximum deflections and stresses in tapered beams and poles.
Normalized tip deflection ($y_A$), maximum stress ($f_{\text{max}}$), and location of maximum stress ($\xi_f$) for a cantilevered tapered beam subjected to:

A - concentrated load at tip; B - uniform load; C - increasing load; 
D - trapezoidal load ($p = r$); E - parabolic load; F - self-weight load.
FIG. 2

Normalized maximum deflection ($y_{max}$) and stress ($f_{max}$) and their locations ($\xi_y$ and $\xi_f$) for a simply supported tapered beam subjected to: G - midspan concentrated load; H - moving concentrated load; I - uniform load; J - increasing load.
Normalized maximum deflection \((y_{\text{max}})\) and stress \((f_{\text{max}})\) and their locations \((\xi_y\) and \(\xi_f\)) for a simply supported tapered beam subjected to:
K - trapezoidal load \((p = r)\); L - parabolic load; M - self-weight load.