FINITE ELEMENT TECHNIQUES
FOR ORTHOTROPIC PLANE
STRESS AND
ORTHOTROPIC PLATE ANALYSIS
ABSTRACT

This paper develops finite element techniques for applicability to plane stress problems and plate problems involving orthotropic materials such as wood and plywood. Applications to limited examples show that the methods have merit especially if means of handling very large systems of equations are utilized.
The solutions of many plane stress problems are literally impossible when attempted by applying the differential equations of the theory of elasticity. For the solution of such problems, stress analysts have sought other methods. One of these methods has been termed the finite element technique, and appears to have merit in yielding approximate solutions to such problems.

The basic concept of the method consists of replacing the solid elastic body to be analyzed by a network of finite elements. It is believed that as the size of the finite element approaches the differential element stage, the results yielded by the method would compare favorably to those obtained from a rigorous mathematical analysis. By keeping the element finite in size, the network model would no longer yield equivalent results but should represent a close approximation.

The finite element technique of plane stress analysis has been presented in different papers by Hrennikoff (2), McCormick (4), Turner (6), and Melosh (5) to name a few. The technique has gained considerable recognition with application to problems associated with the aircraft industry. In all of these papers, however, the technique has been applied to problems associated with isotropic materials. In general, the technique can be divided into two separate subcategories: (1) The framework method and (2) the stiffness element method. The two methods differ principally in the composition of the finite element. It is the purpose of this paper to examine each of these methods closely and determine their applicability in handling problems in orthotropic plane stress.

1Maintained at Madison, Wis., in cooperation with the University of Wisconsin.
2Underlined numbers in parentheses refer to Literature Cited at the end of this Paper.
METHODS

Framework Method

The framework method is appropriately named since the method consists of replacing the solid elastic body to be analyzed by a mathematical model of an imaginary framework. The framework is composed of a series of pin-connected bars arranged in a definite pattern compatible with the type of problem solved, such as the cantilever beam of figure 1.

The most common pattern used in the framework method, and the one used by McCormick, is a rectangular element with diagonals connecting the corners, as shown in figure 2.

\[ \mu = \frac{1}{3} \]

where \( \mu \) is Poisson's ratio of the solid elastic body.

The physical significance of this has not been established, but the proximity of \( \frac{1}{3} \) to the actual value of Poisson's ratio for isotropic materials permits the applicability of the method for such materials.

In orthotropic plane stress, however, there are two Poisson's ratios associated with stress in the \( x-y \) plane. This could perhaps be seen more clearly by observing the form of the generalized stress-strain relationships for orthotropic plane stress:

\[ \epsilon_x = \frac{\sigma_x}{E_x} - \mu_{yx} \frac{\sigma_y}{E_y} \] (1)
\[ \varepsilon_y = \frac{\sigma_y}{E_y} - \mu_{xy} \frac{\sigma_x}{E_x} \]

\[ \varepsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}} \]

where \( \mu_{yx} \) and \( \mu_{xy} \) are the two Poisson's ratios associated with the x-y plane and where \( \varepsilon_x, \varepsilon_y, \) and \( \varepsilon_{xy} \) are the strains associated with the x, y, and x-y directions, respectively. This might imply an even greater restriction of the framework method when applied to orthotropic materials. Computations will now be carried out similar to those performed for isotropic material by Hrennikoff and McCormick, to determine the restriction in orthotropic plane stress.

Mathematical analysis.--Consider a solid orthotropic elastic element (wood) of thickness \( t \), and the corresponding framework model subjected to a situation of pure shear stress:
For such a stress situation, since $\epsilon_x \neq 0, \epsilon_y = 0, \epsilon_z = 0$ (where $\epsilon$ is strain), the diagonal members must supply the shear stiffness. The required stiffness ($S_d = A_d E_d$) of the diagonal members can be determined by considering equilibrium of a section as:

\[
\begin{align*}
\sigma_2 &= \frac{at}{2} \sigma_{xy} \\
\frac{kat}{2} \sigma_{xy} &= \frac{P_1}{P_2} \\
\end{align*}
\]

where:
\[
\begin{align*}
\sin \alpha &= \frac{k}{\sqrt{k^2 + 1}} \\
\cos \alpha &= \frac{1}{\sqrt{k^2 + 1}} \\
\end{align*}
\]

since: $\epsilon_x = \epsilon_y = 0$

therefore: $P_1 = P_3 = 0$

Therefore:
\[
P_2 = \frac{at \sigma}{2} \sqrt{k^2 + 1}
\] (2)

From Mohr’s circle (see Appendix I):
\[
\epsilon_2 = \epsilon_{xy} \sin \alpha \cos \alpha = \epsilon_{xy} \left( \frac{k}{k^2 + 1} \right)
\] (3)

Also:
\[
\epsilon_2 = \frac{P_2}{S_d} = \frac{at \sigma}{2 S_d} \sqrt{k^2 + 1}
\] (4)

equating (3) and (4):
\[
S_d = \frac{at \sigma}{2 \epsilon_{xy}} \left( \frac{k^2 + 1}{k} \right)^{3/2}
\] (5)

From consideration of deformability of the solid element:
\[
\epsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}}
\] (6)

Substituting (6) in (5) results in:
\[
S_d = \frac{at G_{xy}}{2 \epsilon_{xy}} \left( \frac{k^2 + 1}{k} \right)^{3/2}
\] (7)
Consider now the stress situation existing such as to create a case of strain in the $x$-direction only, i.e.:

\[ \varepsilon_x \neq 0 \]
\[ \varepsilon_y = 0 \]
\[ \varepsilon_{xy} = 0 \]  \hspace{1cm} (8)

Therefore for the elastic body and using equations (1):

\[ \sigma_y = \mu \frac{E_y}{E_x} \sigma_x = \mu \frac{E}{E_x} \sigma_x \]  \hspace{1cm} (9)

(utilizing the reciprocal relationship for orthotropic materials \( \mu \frac{E}{E_x} = \mu \frac{E_y}{E_y} \)).

The expression for \( \sigma_x \) can then be written:

\[ \sigma_x = \frac{E}{\lambda} \varepsilon_x \]  \hspace{1cm} (10)

where \( \lambda = (1 - \mu \frac{E}{E_x}) \).

A pictorial representation of this stress situation would look like:
Considering again the equilibrium of a framework section under this situation:

\[
\frac{kat}{2} \mu_y x \sigma_x
\]

The forces \( P_1 \) and \( P_2 \) can also be determined by realizing that:

\[
P_1 + P_2 \sin \sigma = \frac{\sigma_{at}}{2}
\]
\[
P_2 \cos \sigma = \left( \frac{\mu_y x}{2} \right) \sigma_x
\]

or

\[
P_1 = \frac{x}{2} \left( 1 - k \mu_y \tan \alpha \right) = \frac{x}{2} \left( 1 - k^2 \mu_y \right)
\]

\[
P_2 = \frac{x}{2} \left( k \mu_y \sqrt{k^2 + 1} \right)
\]

The forces \( P_1 \) and \( P_2 \) can also be determined by realizing that:

\[
\epsilon = \epsilon_x
\]

Therefore:

\[
P_1 = S_H \epsilon_1 = S_H \epsilon_x = S_H \frac{\lambda}{E} \sigma_x
\]

Equating (11) and (13) results in:

\[
S_H = \frac{at}{2} \frac{E}{\lambda} \left( 1 - k^2 \mu_y \right)
\]

Also from Mohr's circle it can be seen:

\[
\epsilon_2 = \epsilon_x \sin^2 \alpha
\]
Therefore:

\[ P_2 = S \cdot d \cdot 2 \cdot \sin^2 \alpha = S \cdot \frac{\lambda \sigma_x}{E_x} \cdot \sin^2 \alpha \]  \hspace{1cm} (16)

since \( \sin^2 \alpha = \frac{k^2}{k^2 + 1} \);

\[ P_2 = S \cdot \frac{\lambda \sigma_x}{E_x} \left( \frac{k^2}{k^2 + 1} \right) \]  \hspace{1cm} (17)

equating (12) and (17) results in:

\[ S_d = \frac{st}{2} \left( \frac{E_x}{\lambda} \right) \frac{\mu_{yx}}{\lambda} \left( \frac{k^2 + 1}{k} \right)^{3/2} \]  \hspace{1cm} (18)

The same expression for \( S_d \) in (18) could be obtained using a stress situation such that \( \epsilon_y \neq 0 \), \( \epsilon_x = 0 \), \( \epsilon_{xy} = 0 \). The point is, however, that it is also the function of the diagonal members to provide stiffness for the Poisson’s effect when the model is subjected to extensional strain only. The stiffness \( (S_d) \) as given by (18) and (7) must be equal. Hence, equating, we have:

\[ \mu_{yx} = \frac{G_{xy}}{E_x} \]  \hspace{1cm} (19)

which in general is not true for orthotropic materials. In view of this basic ingenerality, therefore, it is concluded that the framework method is not applicable to general orthotropic plane stress problems, but should give good results for the special case of orthotropy as defined by equation (19).

Stiffness Element Method

The stiffness element method differs from that of the framework method in that the elements in the network system are solid or plate elements, and further the elastic properties of the element should duplicate the material it replaces. The elements still remain connected to each other only at the corners or nodes, This perhaps can be visualized more easily if reference is made to a typical orthotropic beam problem shown in figure 3 which will also be the subject of the following discussion on orthotropic plane stress. From figure 3 it is easy to visualize that as the element size decreases, or the number of elements increases, the behavior of the beam model will tend to approach the true behavior of the orthotropic beam.
The stress situation in the beam model will be determined by the manner in which the forces are propagated from element node to element node. For any one element the forces are related to the displacement of the element nodes, since deformability is the physical feature determining the manner of stress propagation through the stressed medium. Each node of the network may have external forces applied in the \( x \) and \( y \) directions (coordinate system as shown in fig. 4). The deformation of a single element, of thickness \( t \), such as shown in figure 4 is defined by the eight possible nodal displacements.

The relationships between the forces and displacements of a single element can be conveniently handled in matrix form as:

\[
\{f\} = [K]_{8\times8} \{\theta\}_{8\times1}
\]

where \( \{\} \), \( [\] \) indicate a column and rectangular matrix, respectively.

The \( [K] \) matrix in equation (20) is generally termed the stiffness matrix, hence the name given to this method.

In this method the structure is regarded as an assemblage of parts and each component has associated with it a stiffness matrix relating the forces and displacements at its nodes. The stiffness matrix for the complete connected structure is then obtained by addition of all the component stiffness matrices. For a major portion of plane stress problems and those to be dealt with in this paper, the object to be analyzed is considered to be homogeneous throughout, which means each component stiffness matrix is identical.

**SPECIFIC OBJECTIVES**

The remaining objectives of this paper, therefore, are to: A. Develop the stiffness \( [K] \) matrix for a single orthotropic element. B. Determine a technique in which the component stiffness matrices can be conveniently handled without overflowing available computer capacity. C. Check these results by investigating simple problems in which the stress distributions are well established. D. Develop
a stiffness $[K]$ matrix which might be used in orthotropic plate analysis. And finally, E. Check the results of the bending stiffness matrix by analyzing a hypothetical orthotropic plate and comparing the results with a rigorous mathematical analysis.

**Develop K Matrix for Single Element**

Consider for analysis the orthotropic plate element as shown in figure 4 with the forces and displacements shown in their positive directions. The stresses and strains for such an element are related by:

$$
\begin{align*}
\epsilon_x &= \frac{\sigma_x}{E_x} - \mu \frac{\sigma_y}{E_y} \\
\epsilon_y &= \frac{\sigma_y}{E_y} - \mu \frac{\sigma_x}{E_x} \\
\epsilon_{xy} &= \frac{\sigma_{xy}}{G_{xy}}
\end{align*}
$$

(21)

The coefficients in any one column of the $K$ matrix represent physically the forces which must be applied at the nodes in order to give a displacement of unity for the particular column chosen while the remaining displacements remain zero. It is this part of the derivation which determines the number of strain expressions or alternatively the number of applied stress states which must be used to achieve this. The number is always twice the number of nodes minus three. Hence, for a rectangular element, five states are required, which will be:
For ease in derivation, the principle of superposition will be utilized so that each case may be handled separately and the results added to determine the combined effect.

Consider first:

(a) Strain in the \( x \)-direction, i.e., \( \epsilon_x \neq 0, \epsilon_y = 0, \epsilon_{xy} = 0 \)

From equations (21), the stresses for an orthotropic element are found to be:

\[
\begin{align*}
\sigma_x &= \frac{E}{\lambda} \epsilon_x \\
\sigma_y &= \mu \epsilon_x \\
\sigma_{xy} &= 0
\end{align*}
\]

It is assumed, however, that these stresses can be replaced by the following equivalent forces applied at the nodal points (refer to fig. 4 for sign convention):

\[
-k_1 = -k_3 = f_5 = f_7 = \frac{\sigma_{X}}{2} = \frac{RLt}{2} \frac{E}{\lambda} \epsilon_x
\]

The expression for \( E_X \) can be written:

\[
\epsilon_x = \frac{(\delta_5 + \delta_7) - (\delta_1 + \delta_3)}{2L}
\]

Substituting, therefore:

\[
-k_1 = -k_3 = f_5 = f_7 = \frac{RLt}{4} \frac{E}{\lambda} \left(-\delta_1 - \delta_3 + \delta_5 + \delta_7\right)
\]

Also:

\[
f_2 = f_6 = -f_4 = -f_8 = \frac{\sigma_{Y}}{2} = \frac{Llt}{2} \frac{E}{\mu \lambda} \epsilon_y
\]

or

\[
f_2 = f_6 = -f_4 = -f_8 = \frac{t}{4} \mu \frac{E}{\lambda} \epsilon_y
\]

Consider next:

(b) Strain in the \( y \)-direction, i.e., \( \epsilon_y \neq 0, \epsilon_x = 0, \epsilon_{xy} = 0 \)

The stresses induced, therefore, are:

\[
\sigma_y = \frac{E}{\lambda} \epsilon_y
\]
The equivalent force system, therefore:

\[
\begin{align*}
\sigma_x &= \frac{E_y}{\lambda} \gamma_x \\
\sigma_y &= \frac{E_y}{\lambda^2} \gamma_y \\
\sigma_{xy} &= 0
\end{align*}
\]

where:

\[
\gamma_y = \frac{(\delta_2 + \delta_6) - (\delta_4 + \delta_8)}{2RL}
\]

Then:

\[
f_2 = f_6 = -f_4 = -f_8 = \frac{tL}{4R} \gamma\left(\delta_2 - \delta_4 + \delta_6 - \delta_8\right)
\]

Also:

\[
-f_1 = -f_3 = f_5 = f_7 = \frac{RLt}{2} \mu \gamma \frac{E_y}{\lambda^2} \\
-f_1 = -f_3 = f_5 = f_7 = \frac{t}{4R} \mu \gamma \left(\delta_2 - \delta_4 + \delta_6 - \delta_8\right)
\]

(c) Shear Strain, i.e., \( \epsilon_{xy} \neq 0, \epsilon_x = 0, \epsilon_y = 0 \)

The expression for shear stress is given by:

\[
\sigma_{xy} = G_{xy} \epsilon_{xy}
\]

Considering forces on the plate element:

![Diagram](image-url)
from figure 5, therefore, it can be seen that:

\[ \varepsilon_{xy} = \frac{(\delta_6 - \delta_2) + (\delta_8 - \delta_4)}{2L} = \frac{(\delta_5 - \delta_7) + (\delta_1 - \delta_3)}{2RL} \]

In equivalent force system

\[ f_1 = f_5 = -f_3 = -f_7 = \frac{Lt}{2} \sigma_{xy} = \frac{Ll}{2} G \varepsilon_{xy} = \frac{G t}{4R} (\delta_1 - \delta_3 + \delta_5 - \delta_7) \]

and also:

\[ f_1 = f_5 = -f_3 = -f_7 = \frac{G t}{4} (-\delta_2 - \delta_4 + \delta_6 + \delta_8) \]

Similarly:

\[ -f_2 = -f_4 = f_6 = f_8 = \frac{RLt}{2} \sigma_{xy} = \frac{RLt}{2} G \varepsilon_{xy} = \frac{G t}{4} (\delta_1 - \delta_3 + \delta_5 - \delta_7) \]

and

\[ -f_2 = -f_4 = f_6 = f_8 = \frac{RG t}{4} (-\delta_2 - \delta_4 + \delta_6 + \delta_8) \]

(d) Pure Bending About x-Axis

Consider element with bending forces:

Since \( \sigma_y = 0 \), \( \sigma_{xy} = 0 \),

therefore \( \sigma_x = E \varepsilon_x \).

In considering the equivalent force system for this case, care must be taken to insure equilibrium. The typical force relationship becomes:
Therefore:

\[ f_5 = \frac{5}{6} \left( \frac{RLt}{4} \right) x - \frac{1}{6} \left( \frac{RLt}{4} \right) x = \frac{RLt}{6} x = -f_7 \]

or

\[ f_1 = -f_3 = -f_5 = f_7 = \frac{RLt}{6} \sigma = \frac{RLt}{6} E \epsilon x. \]

where:

\[ \epsilon = \frac{(-\delta_5 + \delta_1) + (-\delta_3 + \delta_7)}{2L} \]

Then:

\[ f_1 = -f_3 = -f_5 = f_7 = \frac{RLt}{12} E (\delta_1 - \delta_3 - \delta_5 + \delta_7) \]

(e) Pure Bending About y-Axis

The positive bending action can be represented by:
Since $\sigma_x = 0$, $\sigma_{xy} = 0$

therefore $\sigma_y = E_y \varepsilon_y$

The equivalent force system can be written:

$$-f_2 = f_4 = f_6 = -f_8 = \frac{Lt}{6} E_y \varepsilon_y$$

where

$$\varepsilon_y = \frac{\left( -\delta_2 + \delta_4 \right) + \left( \delta_6 - \delta_8 \right)}{2RL}$$

Then:

$$-f_2 = f_4 = f_6 = -f_8 = \frac{E_y}{12R} \left( -\delta_2 + \delta_4 + \delta_6 - \delta_8 \right)$$

By superimposing the relationships obtained between forces and displacements for the five stress states, the final equations can be written in matrix form as:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \\ f_6 \\ f_7 \\ f_8 \end{bmatrix} = \begin{bmatrix} A & -C & D & G & -H & -G & -J & C \\ -C & B & -G & -L & G & M & C & -N \\ D & -G & A & C & -J & -C & -H & G \\ G & -L & C & B & -C & -N & -G & M \\ -H & G & -J & -C & A & C & D & -G \\ -G & M & -C & -N & C & B & G & -L \\ -J & C & -H & -G & D & G & A & -C \\ C & -N & G & M & -G & -L & -C & B \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \\ \delta_5 \\ \delta_6 \\ \delta_7 \\ \delta_8 \end{bmatrix}$$

where:

$$A = \frac{RtE}{12 \lambda} (3 + \lambda) + \frac{tG_{xy}}{4R}$$

$$B = \frac{tE}{12R \lambda} (3 + \lambda) + \frac{RtG_{xy}}{4}$$

$$C = \frac{t}{4 \mu} \frac{E}{yx} \lambda + \frac{tG_{xy}}{4}$$

$$D = \frac{RtE}{12 \lambda} (3 - \lambda) - \frac{tG_{xy}}{4R}$$

$$G = \frac{t}{4 \mu} \frac{E}{yx} \lambda - \frac{tG_{xy}}{4}$$

$$H = + \frac{RtE}{12 \lambda} (3 + \lambda) - \frac{tG_{xy}}{4R}$$

$$J = + \frac{RtE}{12 \lambda} (3 - \lambda) + \frac{tG_{xy}}{4R}$$
or simply:

\[ f = [K] \{ \delta \} \]

where the \([K]\) matrix represents the matrix of coefficients as given in (22).

In order to obtain numerical results in later calculations, it will be assumed that the orthotropic beam in figure 3 has the following elastic properties:

\[
\begin{align*}
\mu_{xy} &= \frac{2}{5}, & \mu_{yx} &= \frac{1}{40}, & E_y &= \frac{1}{16}, & E_x &= \frac{1}{12}
\end{align*}
\]

Therefore:

\[
\lambda = (1 - \mu_{yx}\mu_{xy}) = 0.99.
\]

It will also be assumed that the finite elements are square so that:

\[
R = 1.
\]

The values of the coefficients then become:

\[
\begin{align*}
A &= 0.356691919(E_x); & H &= 0.315025252(E_t) \\
B &= 0.041824495(E_x); & J &= 0.190025252(E_t) \\
C &= 0.027146465(E_x); & L &= 0.000157828(E_t) \\
D &= 0.148358586(E_x); & M &= -0.010258838(E_t) \\
G &= -0.014520202(E_t); & N &= 0.031407828(E_t)
\end{align*}
\]

Therefore the \([K]\) matrix becomes:

\[
\begin{bmatrix}
0.35669 & -0.02715 & 0.14836 & -0.01452 & -0.31503 & 0.01452 & -0.19003 & 0.02715 \\
-0.02715 & 0.04182 & 0.01452 & -0.00016 & -0.01452 & -0.01026 & 0.02715 & -0.03141 \\
0.14836 & 0.01452 & 0.35669 & 0.02715 & -0.19003 & -0.2715 & -0.31503 & -0.01452 \\
-0.01452 & -0.00016 & 0.02715 & 0.04182 & -0.02715 & -0.03141 & 0.01452 & -0.01026 \\
-0.31503 & -0.01452 & -0.19003 & -0.02715 & 0.35669 & 0.02715 & 0.14836 & 0.01452 \\
0.01452 & -0.01026 & -0.02715 & -0.03141 & 0.02715 & 0.04182 & -0.01452 & -0.00016 \\
-0.19003 & 0.02715 & -0.31503 & 0.01452 & 0.14836 & -0.01452 & 0.35669 & -0.2715 \\
0.02715 & -0.03141 & -0.01452 & -0.01026 & 0.01452 & -0.00016 & -0.02715 & 0.04182
\end{bmatrix}
\]
where

\[
\{f\} = E t \begin{bmatrix} K \end{bmatrix} \{\delta\}
\]

Develop a Technique for Handling the Component Stiffness Matrices

With the establishment of the component stiffness matrix it is now desirable to formulate a composite stiffness matrix which would relate, for example, all the nodal forces and deformations occurring in our beam model in figure 3. For many problems, however, it is not necessary to model the whole structure where in most places the stress distribution is well established, but only adjacent to locations where irregularities occur which might effect the normal stress distribution. For purposes of verification and discussion, therefore, let us assume it is desirable to model a section of the beam adjacent to the \( \square \), where symmetry might be observed, and use can be made of the boundary condition that horizontal displacement at the \( \square \) must be zero. The beam model of such a section is shown in figure 6.
The size of a network (9 x 9) of elements was limited to the available computer capacity. The elements are numbered vertically and the arrows indicate the direction of positive forces and displacements.

The composite stiffness matrix, then, will relate all the forces to displacements occurring in the modeled section. Such a matrix, however, would represent the solution of 200 simultaneous equations, which might prove too cumbersome for many small digital computers. It is necessary, therefore, to develop a technique by which the matrix size might be kept small.

It is first necessary to build a typical stiffness matrix for the column of elements being modeled. Since the model is assumed to be homogeneous, all such stiffness matrices are identical, the only change being the nodal values. The stiffness matrix, for example, for the first column of elements can be obtained by combining the elemental component matrices one through nine (see fig. 6). This results in a matrix of size 40 x 40, relating the first 40 forces and displacements. The second matrix is identical to the first with the exception it relates the forces and displacements 21 through 60. The composite matrix for the first two columns of elements can he obtained by combining the individual matrices, realizing that forces and displacement 21 through 40 are common to both columns of elements, resulting in a matrix of size 60 by 60. This may be represented in the form of a partitioned matrix as:

\[
\begin{bmatrix}
\mathbf{f}_1 & f_20 \\
\mathbf{f}_{21} & \mathbf{f}_{40} \\
\mathbf{f}_{41} & \mathbf{f}_{60}
\end{bmatrix}
= \begin{bmatrix}
\mathbf{K}_1 & \mathbf{K}_2 & \mathbf{K}_3 \\
\mathbf{K}_4 & \mathbf{K}_5 & \mathbf{K}_6 \\
\mathbf{K}_7 & \mathbf{K}_8 & \mathbf{K}_9
\end{bmatrix}
\begin{bmatrix}
\delta_1 \\
\delta_{20} \\
\delta_{21} \\
\delta_{40} \\
\delta_{41} \\
\delta_{60}
\end{bmatrix}
\tag{23}
\]

(where the \( \mathbf{K}_i \) matrices are of size 20 by 20.)

If the type of problem to be solved is restricted so that no external loading of the structure will occur between extreme sections of the model, then use can be made of the fact that internal force equilibrium must occur at each common node or:

\[
\mathbf{f}_{\text{column 1}} + \mathbf{f}_{\text{column 2}} = 0
\tag{24}
\]
To make use of equation (24), it is first convenient to interchange rows and columns of equations (23) so to read:

\[
\begin{pmatrix}
  f_1 \\
  f_{20} \\
  f_{21} \\
  f_{40}
\end{pmatrix}
\begin{pmatrix}
  K_1 & K_3 & K_2 \\
  K_7 & K_9 & K_8 \\
  K_4 & K_6 & K_5
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  f_{20} \\
  f_{41} \\
  f_{60}
\end{pmatrix}
\]

(25)

or

\[
\begin{pmatrix}
  f_1 \\
  f_{20} \\
  f_{41} \\
  f_{60}
\end{pmatrix}
\begin{pmatrix}
  A_{40 \times 40} & B_{40 \times 20} \\
  C_{20 \times 40} & D_{20 \times 20}
\end{pmatrix}
\begin{pmatrix}
  f_1 \\
  f_{20} \\
  f_{41} \\
  f_{60}
\end{pmatrix}
\]

(26)

where:

\[
[A] = \begin{bmatrix} K_1 & K_3 \\ K_7 & K_9 \end{bmatrix}
\]

\[
[B] = \begin{bmatrix} K_2 \\ K_8 \end{bmatrix}
\]

\[
[C] = \begin{bmatrix} K_4 & K_6 \end{bmatrix}
\]

\[
[D] = \begin{bmatrix} K_5 \end{bmatrix}
\]
Writing equation (24) in matrix form gives:

\[
\begin{bmatrix}
\delta_1 \\
\delta_{20} \\
\delta_{41} \\
\delta_{60}
\end{bmatrix}
\begin{bmatrix}
C
\end{bmatrix}
+ \begin{bmatrix}
\delta_1 \\
\delta_{20} \\
\delta_{41} \\
\delta_{60}
\end{bmatrix}
\begin{bmatrix}
D
\end{bmatrix}
= \{0\}
\]

or

\[
\begin{bmatrix}
\delta_2 \\
\delta_{40}
\end{bmatrix}
= -D^{-1}[C]
\begin{bmatrix}
\delta_1 \\
\delta_{20} \\
\delta_{41} \\
\delta_{60}
\end{bmatrix}
\tag{27}
\]

The forces and displacements 1 through 20 and 41 through 60 can now be written:

\[
\begin{bmatrix}
f_1 \\
f_{20} \\
f_{41} \\
f_{60}
\end{bmatrix}
= [A]
\begin{bmatrix}
\delta_1 \\
\delta_{20} \\
\delta_{41} \\
\delta_{60}
\end{bmatrix}
+ [B]
\begin{bmatrix}
\delta_{21} \\
\delta_{40}
\end{bmatrix}
\tag{28}
\]

Substituting equations (27) in equations (28) results in:

\[
\begin{bmatrix}
f_1 \\
f_{20} \\
f_{41} \\
f_{60}
\end{bmatrix}
= [A]
\begin{bmatrix}
\delta_1 \\
\delta_{20} \\
\delta_{41} \\
\delta_{60}
\end{bmatrix}
- [B][D^{-1}][C]
\begin{bmatrix}
\delta_{21} \\
\delta_{40}
\end{bmatrix}
\]

or finally:

\[
\begin{bmatrix}
f_1 \\
f_{20} \\
f_{41} \\
f_{60}
\end{bmatrix}
= [A - BD^{-1}C]_{40\times40}
\begin{bmatrix}
\delta_{21} \\
\delta_{40}
\end{bmatrix}
\tag{29}
\]
This process of eliminating the common nodal values can be repeated until the desired network size is achieved. For the particular size network chosen here, the final $\mathbf{A}$ matrix is modified so that the final equations become

$$\begin{bmatrix}
\vec{f}_1 \\
\vec{f}_20 \\
\vec{f}_{181} \\
\vec{f}_{200}
\end{bmatrix} = \begin{bmatrix}
\mathbf{A}_{MOD} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}
\end{bmatrix} \begin{bmatrix}
\vec{\delta}_1 \\
\vec{\delta}_20 \\
\vec{\delta}_{181} \\
\vec{\delta}_{200}
\end{bmatrix}$$

(30)

Verifying the Beam Model

With the establishment of the $\mathbf{A}$ modified matrix for the 9 by 9 network, it is now possible to check the model’s ability to duplicate the original beam by subjecting it to stress distributions which are well established and calculate whether the matrix analysis yields equivalent results. For purposes of comparison, therefore, the following three loading conditions will be applied to the beam model in figures 3 and 6; (1) a force $F$ acting alone, such that a tensile stress of $\sigma_x = 9,000$ pounds per square inch is induced, (2) the forces $P$ acting alone such that a maximum bending stress of $\sigma_x = 10,000$ pounds per square inch is induced in the extreme fibers of the beam model, and (3) a combination (1) and (2).

The given stress loading conditions will be applied as a statically equivalent force system derived from states of loading designated as (a) and (d) in the section “Develop $\mathbf{K}$ Matrix for Single Element.” These forces will be applied to the extreme left section of the beam model and then the stress situation calculated at the centroid of each element in the ninth column. The stress distributions will then be compared with the known distribution.

This operation can perhaps be seen more clearly by beginning with the equations:

$$\begin{bmatrix}
\vec{f}_1 \\
\vec{f}_20 \\
\vec{f}_{181} \\
\vec{f}_{200}
\end{bmatrix} = \begin{bmatrix}
\mathbf{A}_{MOD} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}
\end{bmatrix} \begin{bmatrix}
\vec{\delta}_1 \\
\vec{\delta}_20 \\
\vec{\delta}_{181} \\
\vec{\delta}_{200}
\end{bmatrix}$$

(30)
By observing the boundary conditions at the \( C \), i.e., \( \delta_{181} = \delta_{183} = \delta_{185} = \delta_{187} = \delta_{189} = \delta_{190} = \delta_{191} = \delta_{193} = \delta_{195} = \delta_{197} = \delta_{199} = 0 \), the matrix can be reduced to a 29 by 29 and inverted yielding

\[
\begin{bmatrix}
\delta_1 \\
\delta_{20} \\
\vdots \\
\delta_{200}
\end{bmatrix}
= \begin{bmatrix}
n_1 \\
n_{20} \\
\vdots \\
n_{200}
\end{bmatrix}
= \left[ A^{-1} \right]_{29 \times 29}
\]

The \( \{ f_i \} \) matrix now represents the external force matrix which, for the three loading conditions, is presented on page 22, where \( (dt) \) represents the cross-sectional area of the beam.

The displacements in equation (31) can now be determined by carrying out the matrix operation defined, for each of the three cases. The displacement \( \delta_{161} \) through \( \delta_{180} \) can be gotten from node elimination equation (27), which for addition of the ninth column would appear as

\[
\left\{ \begin{array}{c}
\delta_{161} \\
\delta_{180}
\end{array} \right\} = - \left[ D^{-1} \right] \left[ C \right] \left\{ \begin{array}{c}
\delta_1 \\
\delta_{20}
\end{array} \right\}
\]

(\( \text{where the} \ D \ \text{and} \ C \ \text{matrices are determined when the ninth column was added.} \))
<table>
<thead>
<tr>
<th>( f_i )</th>
<th>Tension</th>
<th>Bending</th>
<th>Tension and Bending</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
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<td>514.4033</td>
<td>+14.4033</td>
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<tr>
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<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>3</td>
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<td>864.1975</td>
<td>-135.8025</td>
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<tr>
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<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>5</td>
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<td>617.2840</td>
<td>-382.7160</td>
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<tr>
<td>6</td>
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<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>7</td>
<td>-1.000.0</td>
<td>370.3704</td>
<td>-629.6296</td>
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<tr>
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<td>0.0</td>
<td>0.0</td>
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<tr>
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<td>123.4568</td>
<td>-876.5432</td>
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<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
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<td>-1.000.0</td>
<td>-123.4568</td>
<td>-1.123.4568</td>
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<tr>
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<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>13</td>
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<td>-370.3704</td>
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<tr>
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<td>0.0</td>
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<tr>
<td>15</td>
<td>-1.000.0</td>
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<tr>
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<td>-1.000.0</td>
<td>-864.1975</td>
<td>-1.864.1975</td>
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<tr>
<td>18</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>19</td>
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<td>-514.4033</td>
<td>-1.014.4033</td>
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<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
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<tr>
<td>184</td>
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<td>186</td>
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<tr>
<td>188</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
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<tr>
<td>192</td>
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<td>0.0</td>
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<tr>
<td>194</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>196</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>198</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>200</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>
With the solution of equations (31), all the nodal displacements for column 9 are known. It is now a relatively easy operation to relate these displacements to corresponding stresses at the centroids of each element, by making use of the equations already established in the section “Develop K Matrix for Single Element.”

It is found that the stress matrix for any one element of such a 9 element column can be written:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\sigma_{xy}
\end{bmatrix} = \frac{9}{2d} \begin{bmatrix}
\frac{E_x}{\lambda} \\
\frac{E_y}{\lambda} \\
\frac{E_{xy}}{\lambda}
\end{bmatrix} [S] \{\delta_i\}
\]

where:

\[
[S] = \begin{bmatrix}
-1 & \mu \frac{E_y}{E_x} & -1 & \mu \frac{E_y}{E_x} & 1 & \mu \frac{E_y}{E_x} & 1 & \mu \frac{E_y}{E_x} \\
\mu \frac{E_y}{E_x} & 1 & \mu \frac{E_y}{E_x} & -1 & \mu \frac{E_y}{E_x} & 1 & \mu \frac{E_y}{E_x} & 1 \\
-\frac{1}{E_x} & \frac{1}{E_x} & -\frac{1}{E_x} & \frac{1}{E_x} & \frac{1}{E_x} & -\frac{1}{E_x} & \frac{1}{E_x} & -\frac{1}{E_x} \\
-\frac{1}{E_x} & \frac{1}{E_x} & -\frac{1}{E_x} & \frac{1}{E_x} & \frac{1}{E_x} & -\frac{1}{E_x} & \frac{1}{E_x} & -\frac{1}{E_x} \\
-\frac{\lambda}{E_x} & \frac{\lambda}{E_x} & -\frac{\lambda}{E_x} & \frac{\lambda}{E_x} & \frac{\lambda}{E_x} & -\frac{\lambda}{E_x} & \frac{\lambda}{E_x} & -\frac{\lambda}{E_x} \\
\frac{\lambda}{E_x} & -\frac{\lambda}{E_x} & \frac{\lambda}{E_x} & -\frac{\lambda}{E_x} & \frac{\lambda}{E_x} & -\frac{\lambda}{E_x} & \frac{\lambda}{E_x} & -\frac{\lambda}{E_x}
\end{bmatrix}
\]

where the \(\delta_i\) matrix represents the eight nodal displacements about the element maintaining coordinate values similar to those in figure 4.

The resulting computed stress distributions for column 9 are presented in figure 7. On the basis of this figure, therefore, it is concluded that the finite element technique was successful in modeling these three simple cases of orthotropic plane stress.

Figure 7.--Stress distributions across \(\ell\) of 9th column of orthotropic beam model.
Finite Element Technique in Orthotropic Plate Analysis

Another area in which the finite element technique has gained popularity is in bending problems of thin plates. Most of these problems again have been associated with the aircraft industry, where it seems most of this work has been pioneered. Melosh (5), derives an elemental stiffness matrix utilizing the bending strain energy expression for a uniform flexurally rigid isotropic plate. Most of the derivations performed by Melosh were based on purely geometrical considerations, so that a transformation to derive a plate stiffness matrix for orthotropic thin plates can be accomplished rather easily. For this section, therefore, an elemental stiffness matrix will be derived for orthotropic thin plates, for which the procedure will parallel that of Melosh with the exception of maintaining orthotropic behavior.

Mathematical analysis.--It will be assumed that a given rectangular orthotropic (wood or plywood) plate, as shown in figure 8a, can be modeled by an assemblage of rectangular elements connected at their nodal points, as in figure 8b.

Each element in figure 8b is assigned a stiffness matrix relating the forces and displacements at its nodal points. Each node will have three degrees of freedom, the angles of rotation $\theta$ and $\phi$ about the $x$ and $y$ axis, respectively, and the lateral displacement of each node, $w_i$. Associated with these displacements are the forces $M_{\theta}$, $M_{\phi}$, and $F_i$ at each node, respectively. The stiffness matrix for each

![Typical rectangular plate](image)

![Finite element plate model](image)

Figure 8

element will be of size 12 by 12. The stiffness matrix of the complete structure can then be determined by addition of the individual component matrices.

The bending strain energy expression for a uniform orthotropic plate in which the axes of orthotropy coincide with the plate boundaries is given by (1):

$$ V = \frac{D}{2} \iint \left\{ \left( \frac{\partial^2 w}{\partial x^2} \right)^2 + \alpha \left( \frac{\partial w}{\partial y} \right)^2 + 2\mu \left( \frac{\partial w}{\partial x} \right) \left( \frac{\partial w}{\partial y} \right) + 4\beta \left( \frac{\partial w}{\partial x} \right)^2 \right\} \, dx \, dy $$

(32)

where

$$ D = \frac{E \, h^3}{2\lambda}, \quad \alpha = \frac{E}{E_x}, \quad \beta = \frac{G \, \lambda}{E_x}; \quad \text{and} \quad \lambda = 1 - \mu \left( \frac{E_y}{E_x} \right) $$

FPL 87 24
The stiffness matrix will be developed by adding matrices reflecting the stiffness of each term in the bending energy expression, since each of these terms can be treated separately.

In deriving the plate stiffness matrix, it will be assumed that the bending curvature along the edges of the plate can be expressed in terms of a third-order polynomial, or

\[
\mathbf{w} = A_3 x^3 + A_2 x^2 + A_1 x + A_0
\]  

(33)

where \( \mathbf{w} \) now defines the displacements; for example, along edge 1-3 in Figure 9.

The four constants in expression (33) can be determined by applying the boundary conditions which must exist along that edge; for example, along edge 1-3, the conditions are:

At \( x = 0 \);
\[
\begin{align*}
\mathbf{w} &= \mathbf{w}_1 \\
\frac{d\mathbf{w}}{dx} &= \theta_1
\end{align*}
\]

and at \( x = a \);
\[
\begin{align*}
\mathbf{w} &= \mathbf{w}_3 \\
\frac{d\mathbf{w}}{dx} &= \theta_3
\end{align*}
\]
where it can be found that:

\[
\begin{align*}
A_0 &= w_1 \\
A_1 &= \theta_1 \\
A_2 &= \frac{3}{a^2}(w_3 - w_1) - \frac{3}{a}\theta_1 - \frac{1}{a}(\theta_3 - \theta_1) \\
A_3 &= \frac{1}{a}(\theta_3 - \theta_1) + \frac{2}{a}\theta_1 - \frac{2}{a^2}(w_3 - w_1)
\end{align*}
\]

From these expressions, a relationship for \( \left( \frac{\partial^2 w}{\partial x^2} \right) \) along 1-3 can be written in matrix notation as:

\[
\left( \frac{\partial^2 w}{\partial x^2} \right)_{1-3} = \frac{2}{a} \left[ \begin{array}{c}
\frac{3x}{a} - 2 \\
\frac{6x}{a} - \frac{3}{a} \\
\frac{3x}{a} - 1 \\
\frac{6x}{a} + \frac{3}{a}
\end{array} \right] \begin{bmatrix}
\theta_1 \\
\theta_3 \\
\theta_3 \\
w_3
\end{bmatrix}
\]

(34)

The total strain energy resulting from \( \left( \frac{\partial^2 w}{\partial x^2} \right) \) can be found by squaring expression (34) and integrating the result over half the plate area. (It is also assumed that expression (33) varies uniformly with \( y \).) The force-displacement relationships can then be found for this portion of the plate by utilizing Castigliano's complementary relation:

\[
\frac{\partial V}{\partial \delta_1} = F_1
\]

in which \( \delta_1 \) and \( F_1 \) represent displacement and force components, respectively. Performing these operations then yields the following expressions for edge 1-3:

\[
\begin{align*}
\begin{bmatrix}
M_{\theta_1} \\
F_1 \\
M_{\theta_3} \\
F_3
\end{bmatrix} &= Db \begin{bmatrix}
2a & 3 & a & -3 \\
3 & 6 & a & 3 \\
\frac{2}{a} & 3 & 2a & -3 \\
-3 & \frac{6}{a} & -3 & \frac{6}{a}
\end{bmatrix} \begin{bmatrix}
\theta_1 \\
w_1 \\
\theta_3 \\
w_3
\end{bmatrix}
\end{align*}
\]

(35)

---

\(^3\)The subscript on the differential terms denotes the edge along which the relationship applies.
A similar set of expressions can be found by treating edge 2-4 and using the remaining half of the plate. By adding the two results, the final bending stiffness matrix due to the term \( \frac{D}{2} \left( \frac{\partial^2 w}{\partial x^2} \right)^2 \) in equation (32) can be written:

\[
\begin{bmatrix}
M_{01} \\
F_1 \\
M_{02} \\
F_2 \\
M_{03} \\
F_3 \\
M_{04} \\
F_4
\end{bmatrix} = D
\begin{bmatrix}
2\frac{b}{a} & \frac{3}{2} \frac{b}{a} & 0 & 0 & \frac{b}{a} & -\frac{3}{2} \frac{b}{a} & 0 & 0 \\
\frac{3}{2} \frac{b}{a} & \frac{6}{3} \frac{b}{a} & 0 & 0 & \frac{3}{2} \frac{b}{a} & -\frac{6}{3} \frac{b}{a} & 0 & 0 \\
0 & 0 & \frac{2}{a} \frac{b}{a} & \frac{3}{2} \frac{b}{a} & 0 & 0 & \frac{2}{a} \frac{b}{a} & -\frac{3}{2} \frac{b}{a} \\
0 & 0 & \frac{3}{2} \frac{b}{a} & \frac{6}{3} \frac{b}{a} & 0 & 0 & \frac{3}{2} \frac{b}{a} & -\frac{6}{3} \frac{b}{a} \\
\frac{b}{a} & \frac{3}{2} \frac{b}{a} & 0 & 0 & \frac{2}{a} \frac{b}{a} & -\frac{3}{2} \frac{b}{a} & 0 & 0 \\
0 & 0 & \frac{b}{a} & \frac{3}{2} \frac{b}{a} & 0 & 0 & \frac{b}{a} & -\frac{3}{2} \frac{b}{a} \\
0 & 0 & -\frac{3}{2} \frac{b}{a} & -\frac{6}{3} \frac{b}{a} & 0 & 0 & -\frac{3}{2} \frac{b}{a} & \frac{6}{3} \frac{b}{a} \\
0 & 0 & -\frac{6}{3} \frac{b}{a} & -\frac{6}{3} \frac{b}{a} & 0 & 0 & -\frac{6}{3} \frac{b}{a} & \frac{6}{3} \frac{b}{a}
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
w_1 \\
\theta_2 \\
w_2 \\
\theta_3 \\
w_3 \\
\theta_4 \\
w_4
\end{bmatrix}
\]
If it is assumed that displacements along the edge in the $Y$-direction are of the same general form as given by expression (33), then the bending stiffness matrix for the term can be written by observing symmetry as:

$$
\begin{align*}
M_{\phi_1} &= \begin{bmatrix}
\frac{\alpha a}{b} & \frac{\alpha a}{b} & \frac{\alpha a}{b} & \frac{\alpha a}{b} & 0 & 0 & 0 & 0 \\
\frac{\alpha a}{b^2} & \frac{\alpha a}{b^2} & \frac{\alpha a}{b^2} & \frac{\alpha a}{b^2} & 0 & 0 & 0 & 0 \\
\frac{\alpha a}{b} & \frac{\alpha a}{b} & \frac{\alpha a}{b} & \frac{\alpha a}{b} & 0 & 0 & 0 & 0 \\
\frac{-\alpha a}{b^3} & \frac{-\alpha a}{b^3} & \frac{-\alpha a}{b^3} & \frac{-\alpha a}{b^3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{\alpha a}{b^2} & \frac{\alpha a}{b^2} & -\frac{\alpha a}{b^2} \\
0 & 0 & 0 & 0 & 0 & \frac{\alpha a}{b^2} & \frac{\alpha a}{b^2} & -\frac{\alpha a}{b^2} \\
0 & 0 & 0 & 0 & 0 & \frac{\alpha a}{b^2} & \frac{\alpha a}{b^2} & -\frac{\alpha a}{b^2} \\
0 & 0 & 0 & 0 & 0 & \frac{-\alpha a}{b^3} & \frac{-\alpha a}{b^3} & \frac{-\alpha a}{b^3} & \frac{\alpha a}{b^3} \\
\end{bmatrix} \left( \phi_1 \right) \\
F_1 &= \begin{bmatrix}
\phi_1 \\
w_1 \\
\phi_2 \\
w_2 \\
\phi_3 \\
w_3 \\
\phi_4 \\
w_4 \\
\end{bmatrix}
\end{align*}
\] (37)

To obtain the bending strain energy due to the bending coupling effect, it is necessary to perform the integration:

$$
V = \frac{D}{2} \int \int 2\mu \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \, dx \, dy
$$

over the area of the plate. This can be accomplished, as outlined by Melosh, by taking the product, i.e., \( \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} \) at each node and integrating over only a quadrant of the plate at a time. The sum of the integral of the four products then represents the coupling energy. For example, to obtain the energy for the first quadrant, it is necessary to perform the integration:

$$
V \bigg|_{\text{quadrant 1}} = \mu \frac{D}{2} \int_0^b \int_0^a \frac{\partial w}{\partial x} \left( \frac{\partial^2 w}{\partial x^2} \right)_{1-3} \left( \frac{\partial^2 w}{\partial y^2} \right)_{1-2} \, dx \, dy
$$

where the product \( \left( \frac{\partial^2 w}{\partial x^2} \right)_{1-3} \left( \frac{\partial^2 w}{\partial y^2} \right)_{1-2} \) can be written in matrix form as:
\[
\left( \begin{array}{c}
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial y^2}
\end{array} \right)_{1-3} \left( \begin{array}{c}
\frac{\partial^2 w}{\partial x^2} \\
\frac{\partial^2 w}{\partial y^2}
\end{array} \right)_{1-2}
\]

\[
\begin{bmatrix}
\phi_1 \\
\omega_1 \\
\phi_2 \\
\omega_2
\end{bmatrix} =
\begin{bmatrix}
\theta_1 & 0 & 0 & 0 \\
0 & \omega_1 & 0 & 0 \\
0 & 0 & \theta_3 & 0 \\
0 & 0 & 0 & \omega_3
\end{bmatrix}
\begin{bmatrix}
\frac{3x}{a} - 2 & \frac{3y}{b} - 2 & \frac{6x}{a} - 3 & \frac{6y}{b} - 3 \\
\frac{3x}{a} - 2 & \frac{3y}{b} - 1 & \frac{3x}{a} - 2 & \frac{6y}{b} + 3 \\
\frac{6x}{a} - 3 & \frac{3y}{b} - 2 & \frac{6x}{a} - 3 & \frac{6y}{b} - 3 \\
\frac{3x}{a} - 1 & \frac{3y}{b} - 2 & \frac{6x}{a} - 3 & \frac{3y}{b} - 1
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\phi_2 \\
\phi_2 \\
\phi_2
\end{bmatrix}
\]

Integrating, it can be found that:

\[
V_{\text{quadrant 1}} = \frac{ab}{4} \frac{D}{16^\mu} \begin{bmatrix}
\theta_1 & 0 & 0 & 0 \\
0 & \omega_1 & 0 & 0 \\
0 & 0 & \theta_3 & 0 \\
0 & 0 & 0 & \omega_3
\end{bmatrix}
\begin{bmatrix}
25 & \frac{30}{b} & 5 & \frac{30}{b} \\
30 & \frac{36}{a} & 6 & \frac{36}{b} \\
5 & \frac{6}{a} & 1 & -\frac{6}{b} \\
\frac{30}{a} & \frac{36}{ab} & \frac{-6}{a} & \frac{36}{ab}
\end{bmatrix}
\begin{bmatrix}
\phi_1 \\
\omega_1 \\
\phi_2 \\
\omega_2
\end{bmatrix}
\]

Differentiating with respect to each of the displacement values to obtain the respective force component yields:

\[
\begin{bmatrix}
M_{\theta 1} \\
M_{\phi 1} \\
F_1 \\
M_{\phi 2} \\
F_2 \\
M_{\theta 3} \\
F_3
\end{bmatrix} = \frac{D}{16^\mu} \begin{bmatrix}
0 & 25 & \frac{30}{b} & 5 & \frac{30}{b} & 0 & 0 \\
25 & 0 & \frac{30}{a} & 0 & 0 & 5 & \frac{30}{a} \\
\frac{30}{b} & \frac{30}{a} & \frac{72}{ab} & \frac{6}{a} & \frac{36}{b} & \frac{6}{a} & \frac{36}{b} \\
5 & 0 & \frac{6}{a} & j & 0 & 1 & -\frac{6}{a} \\
\frac{30}{b} & 0 & \frac{36}{ab} & 0 & 0 & \frac{6}{b} & \frac{36}{ab} \\
0 & 5 & \frac{6}{b} & 1 & \frac{6}{b} & 0 & 0 \\
0 & \frac{30}{a} & \frac{36}{ab} & \frac{6}{a} & \frac{36}{ab} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\theta_1 \\
\phi_1 \\
\omega_1 \\
\phi_2 \\
\omega_2 \\
\theta_3 \\
\omega_3
\end{bmatrix}
\]

29
Treating the remaining three quadrants in a similar manner and superimposing the results lead to the following stiffness matrix for the coupling energy term of equation (32):

\[
\begin{bmatrix}
M_{\theta 1} & 0 & 25 & \frac{30}{b} & 0 & 5 & \frac{30}{b} & 0 & -5 & \frac{6}{b} & 0 & -1 & \frac{6}{b} \\
M_{\phi 1} & 25 & 0 & \frac{30}{a} & -5 & 0 & \frac{6}{a} & 5 & 0 & \frac{30}{a} & -1 & 0 & \frac{6}{a} \\
F_1 & 30 & 30 & \frac{72}{b} & \frac{30}{b} & 6 & \frac{6}{b} & \frac{30}{b} & 6 & \frac{6}{b} & 72 & b & a & ab & b & a & ab & b & a & ab \\
M_{\theta 2} & 0 & -5 & \frac{30}{b} & 0 & -25 & \frac{30}{b} & 0 & 1 & \frac{6}{b} & 0 & 5 & \frac{6}{b} \\
M_{\phi 2} & 5 & 0 & \frac{6}{a} & -25 & 0 & \frac{30}{a} & 1 & 0 & \frac{6}{a} & -5 & 0 & \frac{30}{a} \\
F_2 & \frac{30}{b} & \frac{6}{a} & \frac{72}{b} & \frac{30}{b} & \frac{30}{a} & \frac{72}{a} & \frac{6}{a} & \frac{72}{a} & \frac{6}{a} & \frac{72}{a} & \frac{30}{a} & \frac{72}{a} & \frac{72}{a} \\
M_{\theta 3} & 0 & 5 & \frac{6}{b} & 0 & 1 & \frac{6}{b} & 0 & -25 & \frac{30}{b} & 0 & -5 & \frac{30}{b} \\
M_{\phi 3} & -5 & 0 & \frac{30}{a} & 1 & 0 & \frac{6}{a} & -25 & 0 & \frac{30}{a} & 5 & 0 & \frac{6}{a} \\
F_3 & \frac{6}{b} & \frac{30}{a} & \frac{72}{b} & \frac{6}{b} & \frac{6}{a} & \frac{72}{a} & \frac{72}{a} & \frac{30}{a} & \frac{30}{a} & \frac{72}{a} & \frac{30}{a} & \frac{30}{a} & \frac{72}{a} \\
M_{\theta 4} & 0 & -1 & \frac{6}{b} & 0 & -5 & \frac{6}{b} & 0 & 5 & \frac{30}{b} & 0 & 25 & \frac{30}{b} \\
M_{\phi 4} & -1 & 0 & \frac{6}{a} & 5 & 0 & \frac{30}{a} & -5 & 0 & \frac{6}{a} & 25 & 0 & \frac{30}{a} \\
F_4 & \frac{6}{b} & \frac{6}{a} & \frac{72}{b} & \frac{6}{a} & \frac{30}{a} & \frac{72}{a} & \frac{30}{a} & \frac{6}{a} & \frac{72}{a} & \frac{30}{a} & \frac{30}{a} & \frac{72}{a} \\
\end{bmatrix} = \frac{D}{16} \gamma x
\]

\[
\begin{bmatrix}
\theta_1 \\
\phi_1 \\
w_1 \\
\theta_2 \\
\phi_2 \\
w_2 \\
\theta_3 \\
\phi_3 \\
w_3 \\
\theta_4 \\
\phi_4 \\
w_4 \\
\end{bmatrix}
\]

In deriving the stiffness matrix for the torsion term of equation (32), it is necessary to determine its corresponding energy or:

\[
V = \frac{D}{2} \int_{0}^{b} \int_{0}^{a} \frac{2}{4b} \left( \frac{2}{\partial w_{xy}} \right)^2 dx dy
\]  \hspace{1cm} (39)

An expression for \( \frac{\partial^2 w}{\partial x \partial y} \) can be found by considering the simple torsion of the finite element of figure 10, where:

\[
\frac{\partial^2 w}{\partial x \partial y} = \frac{1}{ab} \left( w_1 - w_2 - w_3 + w_4 \right)
\]  \hspace{1cm} (40)
By squaring equation (40), performing the integration of (39), and differentiation as done previously, it can be found that the stiffness matrix for the torsion term can be written:

\[
\begin{bmatrix}
F_1 \\
F_2 \\
F_3 \\
F_4
\end{bmatrix} = \frac{4\beta}{ab} \begin{bmatrix}
1 & -1 & -1 & 1 \\
-1 & 1 & 1 & -1 \\
-1 & 1 & 1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix} \begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
w_4
\end{bmatrix}
\]

(41)

The final stiffness matrix for orthotropic thin plates can now be determined by superposition of expressions (36), (37), (38), and (41), and found to be:

\[
\begin{bmatrix}
M_{\theta 1} \\
M_{\phi 1} \\
F_1 \\
M_{\theta 2} \\
M_{\phi 2} \\
F_2 \\
M_{\theta 3} \\
M_{\phi 3} \\
F_3 \\
M_{\theta 4} \\
M_{\phi 4} \\
F_4
\end{bmatrix} = D
\begin{bmatrix}
2R & 25B & C & 0 & 5B & -E & R & -5B & -d & 0 & -B & \theta_1 \\
25B & (2\frac{\alpha}{R}) & H & -5B & \frac{\alpha}{R} & -K & 5B & 0 & -J & -B & 0 & \phi_1 \\
0 & -5B & -E & 2R & -25B & C & 0 & B & G & R & 5B & \phi_3 \\
5B & \frac{\alpha}{R} & K & -25B & 2\frac{\alpha}{R} & -H & B & 0 & -L & -5B & 0 & \phi_4 \\
R & 5B & d & 0 & B & -G & 2R & -25B & -C & 0 & -5B & \phi_3 \\
-5B & 0 & -J & B & 0 & L & -25B & 2\frac{\alpha}{R} & H & 5B & \frac{\alpha}{R} & \phi_3 \\
0 & -B & -G & R & -5B & d & 0 & 5B & E & 2R & 25B & -C \\
-B & 0 & -L & 5B & 0 & J & -5B & \frac{\alpha}{R} & K & 25B & 2\frac{\alpha}{R} & -H \\
\end{bmatrix}
\]

(42)
where:

\[
\begin{align*}
R &= \frac{b}{a} \\
B &= \frac{1}{16} \mu xy \\
C &= \frac{3}{b} \left( R^2 + 10B \right) \\
d &= \frac{3}{b} \left( R^2 + 2B \right) \\
E &= \frac{3}{b} (10B) \\
G &= \frac{3}{b} (2B) \\
H &= \frac{3}{a} \left( \alpha R^2 + 10B \right) \\
J &= \frac{3}{a} (10B) \\
K &= \frac{3}{a} \left( \alpha R^2 + 2B \right) \\
L &= \frac{3}{a} (2B) \\
A &= \frac{1}{ab} \left( 6R^2 + 8 \frac{\alpha}{R^2} + \frac{9}{2} \mu xy + 4 \beta \right) \\
M &= \frac{1}{ab} \left( 6R^2 + \frac{9}{2} \mu xy + 4 \beta \right) \\
N &= \frac{1}{ab} \left( 6 \frac{\alpha}{R^2} + \frac{9}{2} \mu xy + 4 \beta \right) \\
P &= \frac{1}{ab} \left( \frac{9}{2} \mu xy + 4 \beta \right)
\end{align*}
\]

and recalling that:

\[
D = \frac{E_h^2}{12 \lambda}, \quad \alpha = \frac{E_y}{E_x}, \quad \beta = \frac{G_{xy}}{E_y}, \quad \text{and} \quad \lambda = (1 - \mu_{xy} \mu_{yx})
\]

Where for plywood plates, these elastic properties represent their effective values in the direction denoted by the subscripts. These can be determined experimentally or by knowing the properties of the individual plies, they can be calculated by existing formula (7).

**Comparison of Finite Element Model with Mathematical Analysis**

With the establishment of the stiffness matrix for orthotropic thin plates, it is now desirable to compare the finite element model’s ability to duplicate the behavior of an orthotropic plate as described by a rigorous mathematical solution such as that derived by March (3).

For purposes of comparison, therefore, it will be assumed it is desirable to determine the maximum deflection of the simply-supported five-ply plywood plate of Figure 10 under (1) a concentrated center load \( \mathbf{P} \) and (2) a uniformly distributed load \( q \).
From March it can be found after considerable calculation that the maximum center deflection $W_0$ is given by:

$$W_0 = \frac{6.875 P}{D}$$

for the case of the concentrated load, and by:

$$W_0 = \frac{625.44 \frac{q}{D}}$$

for the case of the uniformly distributed load, where $D$ is as previously defined.

To obtain comparative values by the finite element technique, the plate of Figure 10 will be modeled by 36 finite elements as shown in Figure 11.
By adding the individual elemental stiffness matrices it can be seen that the final composite matrix will be of size 147 by 147. This can be reduced, however, to a 48 by 48 matrix by observing symmetry and analyzing only one quadrant for the particular cases chosen. By applying the appropriate boundary conditions and inverting the resulting matrix, the maximum deflections by the finite element model was found to be:

\[ W_o = 6.800 \frac{P}{D} \]

and

\[ W_o = 599.69 \frac{q}{D} \]

for the cases of the concentrated and uniform loads, respectively.

By comparing results of the two analyses it is seen that for the concentrated load problem the difference is in the neighborhood of 1 percent, while for the uniform load case the difference is approximately 4 percent. The greater error found in the uniform load case is attributed to the fact that in the finite element approach, the uniform load is replaced by a statically equivalent set of concentrated forces acting at the nodal points, and as a result the loads along the boundary of the plate have no effect on the bending of the plate.
CONCLUSIONS

On the basis of the limited examples examined in this report, it appears the finite element technique has merit regarding problems in orthotropic plane stress. It is realized that the networks used in this report were extremely coarse, but this should not distract the fact of the model’s ability to duplicate orthotropic behavior. It is felt that the techniques which need to be developed in the future lie in the effective handling of the large systems of equations that result as finer networks are desired. The primary obstacle possibly is the round-off error occurring in the many digital computations.
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APPENDIX 1

Mohr’s circles of strain for the typical element;

Strain in the x-Direction only.

\[ \varepsilon_2 = \frac{\varepsilon_x}{2} + \frac{\varepsilon_x}{2} \cos(180^\circ - 2\alpha) \]
\[ = \frac{\varepsilon_x}{2} [1 + \cos(180^\circ - 2\alpha)] \]
\[ = \frac{\varepsilon_x}{2} [1 - \cos 2\alpha] \]
\[ = \frac{\varepsilon_x}{2} \sin^2 \alpha \]
Strain in the $y$-Direction only.

Shear Strain in $x$-$y$ Plane.

\[ \varepsilon_2 = \frac{\varepsilon_y}{2} + \frac{\varepsilon_y}{2} \cos(-2a) \]

\[ = \frac{\varepsilon_y}{2}(1 + \cos2a) \]

\[ \varepsilon_2 = \varepsilon_y \cos^2 a \]

\[ \varepsilon_2 = \frac{\varepsilon_{xy}}{2} \sin(180^\circ - 2a) \]

\[ \varepsilon_2 = \frac{\varepsilon_{xy}}{2} \sin2a \]

or \[ \varepsilon_2 = \varepsilon_{xy} \sin a \cos a \]
NOTATION

$\varepsilon_x, \varepsilon_y$  Strains (extension or compression) in the $x$ and $y$ directions, respectively.

$\varepsilon_{xy}$  Strain (shear); the change in angle between lines originally drawn in the $x$ and $y$ directions.

$E_x, E_y$  Modulus of elasticity of wood in the $x$ and $y$ directions, respectively.

$G_{xy}$  Modulus of rigidity associated with shear deformation in the $x$-$y$ plane resulting from shear stresses in the $xz$ and $yz$ planes.

$\sigma_x, \sigma_y$  Normal stress components in the $x$ and $y$ directions, respectively.

$\sigma_{xy}$  Shear stress associated with the $x$-$y$ plane.

$\mu_{xy}$  Poisson's ratio of contraction along the direction $y$ to extension along the direction $x$ due to a normal tensile stress in the direction $x$; similarly, $\mu_{yx}$.

$S_d, S_H$  The stiffness values $(E_iA_i)$ of the diagonal and horizontal member in the framework model.
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