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Two- and Three-Parameter Weibull Goodness-of-Fit Tests

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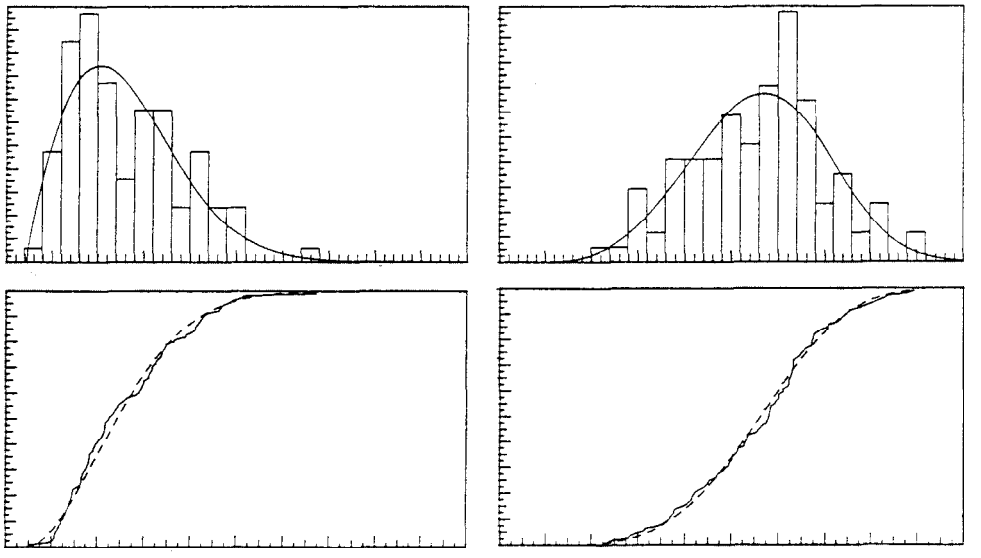

Abstract

Extensive tables of goodness-of-fit critical values for the two- and three-parameter Weibull distributions are developed through simulation for the Kolmogorov-Smirnov statistic, the Anderson-Darling statistic, and Shapiro-Wilk-type correlation statistics. Approximating formulas for the critical, values are derived and compared with values in the tables. Power studies using several different distributional forms show that the Anderson-Darling statistic is the most sensitive to lack of fit of a two-parameter Weibull and correlation statistics of the Shapiro-Wilk type the most sensitive to departures from a three-parameter Weibull.

Keywords: Goodness-of-fit statistics, Weibull distribution, lumber strength, modulus of elasticity

Research Highlights

This paper presents the results of a study to develop and evaluate goodness-of-fit tests for the two- and three-parameter Weibull distributions. The study was initiated because of discrepancies in published critical values for two-parameter Weibull distribution goodness-of-fit tests, the lack of any critical values for a Shapiro-Wilk-type correlation statistic, and the lack of general three-parameter Weibull distribution goodness-of-fit tests. The results of the study will be used by Forest Products Laboratory (FPL) scientists to evaluate the goodness-of-fit of Weibull distributions to experimental data. This will allow evaluation of distributional forms that may be used in reliability-based design procedures.

Through computer simulation, extensive tables of critical values for three standard goodness-of-fit statistics are developed for both the two- and three-parameter Weibull distributions. The statistics used are the Kolmogorov-Smirnov D statistic, the Anderson-Darling A^2 statistic, and a Shapiro-Wilk-type correlation statistic. The critical values for the statistics are modeled through regression to provide equations to estimate the critical values. The equations allow computer programs to evaluate the goodness-of-fit of data sets containing up to 400 observations. The abilities of the tests to detect poor fits (the "power of the tests") are studied using both the equations and the exact critical values of the test statistics. Finally, invariance properties of the statistics are proved. These invariance properties show that the scope of the simulation is adequate for all problems that the tests might be applied to.

Four general conclusions are readily apparent from the results of this study:

1. Of the three statistics considered, the Anderson-Darling A^2 statistic appears to be the statistic of choice for testing the goodness-of-fit of a two-parameter Weibull distribution to a set of data. The choice might be different if censored data were used since the correlation statistic has the potential advantage of being easily modified for type I and type II censored observations.

2. For testing the three-parameter Weibull goodness-of-fit, the correlation statistics R^2_{we} or R^2_{w} appear to be the best.

3. The critical value approximations appear to be very good for the range of sample sizes considered.

4. The power of the tests is very dependent upon the true distributional form of the data.

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Background and Introduction

Two-parameter and three-parameter Weibull distributions are widely used to represent the strength distribution of structural lumber and engineering-designed wood subassemblies. As wood construction practices in the United States and Canada are revised from deterministic to reliability-based design procedures, assessing the goodness-of-fit of these Weibull distributional forms becomes increasingly important. In its three-parameter form, the family is represented by the density function

$$f(x) = ab^{-1}\,[(x - c)/b]^{a-1}\,\exp\{-[(x - c)/b]^{a}\} \qquad (x > c;\ a,\ b > 0) \tag{1}$$

where a is the shape parameter, b the scale parameter, and c the location parameter. The family of two-parameter Weibull distributions follows from Equation (1) when $c = 0$.

Goodness-of-fit tests for the two-parameter Weibull distributions have received considerable attention. Mann and others (1973), Smith and Bain (1976), Stephens (1977), Littell and others (1979), Chandra and others (1981), Tiku and Singh (1981), and Wozniak and Warren (1984) have all discussed aspects of this problem. Mann and others (1973) and Tiku and Singh (1981) proposed new statistics to test the goodness-of-fit of two-parameter Weibull distributions. Smith and Bain (1976) proposed a test statistic analogous to the Shapiro-Francis statistic for testing normality. The Smith and Bain T statistic was based on the sample correlation between the order statistics of a sample and the expected value of the order statistics under the assumption that the sample comes from a two-parameter Weibull distribution. For complete samples and two levels of censoring, they provided critical values for samples

containing 8, 20, 40, 60, or 80 observations. Stephens (1977) produced tables of the asymptotic critical values of the Anderson-Darling A^2 statistic and the Cramer-von Mises W^2 statistic for various significance levels. Littell and others (1979) compared the Mann, Scheuer, and Fertig S statistic, the Smith and Bain T statistic, the modified Kolmogorov-Smirnov D statistic, the modified Cramer-von Mises W^2 statistic, and the modified Anderson-Darling A^2 statistic through a series of power studies for sample size n = 10 to 40. They also calculated critical values for the D, W^2, and A^2 statistics for n = 10, 15, . . . , 40. Chandra and others (1981) calculated critical values for the Kolmogorov-Smirnov D statistic for n = 10, 20, 50, and infinity for three situations.

The three-parameter Weibull distribution has received considerably less attention, probably because the critical values depend upon the unknown shape parameter. Woodruff and others (1983) produced tables of critical values for a modified Kolmogorov-Smirnov test for a Weibull distribution with unknown location and scale parameters and a known shape parameter. Their tables included n = 5 to 15, 20, 25, and 30 for shape parameters of 0.5, 1.0, . . . , 4.0. Bush and others (1983) produced similar tables for the Cramer-von Mises and Anderson-Darling statistics, together with a relationship between the critical value and the inverse shape parameter. No asymptotic results or Monte Carlo determinations for $n > 30$ appear in the literature.

Despite this extensive literature, we encountered difficulties when we studied the goodness-of-fit of two- and three-parameter Weibull distributions to numerous data sets consisting of 80 to 400 observations of various lumber strength properties. For the three-parameter Weibull tests, critical values are not published for the sample sizes involved, and it is not clear how estimated shape parameters would affect the critical values derived by assuming a known shape parameter. For the two-parameter Weibull tests, these difficulties included a lack of published critical values for sample sizes larger than 50 and some apparent inconsistencies in published critical values. These apparent inconsistencies can be seen in our Tables 1 and 2. In Table 1 the Chandra and others (1981) and the Littell and others (1979) critical values for the Kolmogorov-Smirnov D statistic show generally good agreement. (We scaled Littell's values to compare with Chandra's by multiplying the test statistic critical value by $n^{1/2}$.) However, Littell and others (1979) might have obtained a smoother set of values if they had used a larger simulation. Some inconsistencies at the 0.01 level of significance are also evident. Since Littell and others (1979) reported the means of 10 runs of 1,000 simulations and Chandra and others (1981) performed one run of 10,000, the latter values might be expected to be smoother at the 0.01 level. Wozniak and Warren (1984) calculated critical values for the Anderson-Darling statistic for the two-parameter Weibull distribution when parameters are estimated by maximum likelihood. A comparison of their results with those of Stephens (1977) and Littell (Table 2) shows some variability in the second decimal of the critical values. For example, the Stephens critical value at the 0.05 level for n = 40 is the same as the Wozniak and Warren 0.05-level value for n = 15. The Littell value for n = 15 is even larger than the Wozniak value.

In this paper, we (1) develop extensive and definitive two-parameter Weibull distribution goodness-of-fit critical values for the Shapiro-Wilk-type correlation coefficient statistics, the Anderson-Darling A^2 statistic, and the Kolmogorov-Smirnov D statistic, (2) develop and evaluate formulas to yield a smoothed estimate of the critical values for the statistics, and (3) extend these three statistics for use as goodness-of-fit statistics with a three-parameter Weibull distribution.

Shapiro-Wilk-Type Correlation Statistic

Studies such as the Monte-Carlo study of Shapiro and others (1968) have consistently shown that for testing goodness-of-fit of normal distributions, the Shapiro-Wilk statistic has superior power to other statistics in detecting that the data comes from a wide range of other distributions. To develop a similar goodness-of-fit test for Weibull distributions, we modify a simplified form of this statistic first suggested by Shapiro and Francia (1972) but with approximate "scores" suggested by Filliben (1975).

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote an ordered sample of size n from the population of interest. Specifically, for the two- and three-parameter Weibull distribution, we consider the test statistic R_w^2, where

$$R_w = \frac{\sum_{i=1}^{n} (X_{(i)} - \overline{X})\, m_{w,i}}{\left[\sum_{i=1}^{n} (X_{(i)} - \overline{X})^2 \sum_{i=1}^{n} (m_{w,i} - \overline{m})^2\right]^{1/2}} \tag{2}$$

and

$$m_{w,i} = \{-\ln[1 - (i - 0.3175) / (n + 0.365)]\}^{1/\hat{a}}$$

is a median score, in the spirit of Filliben, except that these scores depend upon the maximum likelihood estimate $\hat{a}$ of the Weibull shape parameter a. The corresponding Q-Q plot of $(X_{(i)}, m_{w,i})$ should resemble a straight line if the underlying population is Weibull. (A Q-Q plot is an abbreviated notation for a Quantile-Quantile probability plot, where corresponding percentiles of one distribution are plotted against the percentiles of the other, as discussed in Wilk and Gnanadesikan (1968).) The statistic R_w^2 is the squared correlation coefficient of this plot. Note that the choice of $m_{w,i}$ follows the approximate scores suggested by Filliben (1975) and is slightly different from that chosen by Smith and Bain (1976).

If a variable X has the two-parameter Weibull distribution, the variable $Y = \ln X$ has an extreme value distribution. Calculating goodness-of-fit on this scale has advantages. Since the extreme value distributions are defined by location and scale parameters, the critical values for the correlation statistic are not dependent on the true shape parameter. Thus, for two-parameter Weibull distributions, we propose the correlation-type statistic R_{we}^2, where

$$R_{we} = \frac{\sum_{i=1}^{n} (\ln X_{(i)} - \overline{\ln X}) \ln m_{w,i}}{\left[\sum_{i=1}^{n} (\ln X_{(i)} - \overline{\ln X})^2 \sum_{i=1}^{n} (\ln m_{w,i} - \overline{\ln m})^2\right]^{1/2}} \tag{3}$$

and

$$\ln m_{w,i} = \ln\left\{\left[-\ln\left(1 - \frac{i - 0.3175}{n + 0.365}\right)\right]^{1/\hat{a}}\right\}$$

Other Goodness-of-Fit Statistics

We include two other well-studied goodness-of-fit statistics for comparative purposes. The modified Komogorov-Smirnov D statistic is given by

$$D = \sup |\, F(x;b,a) - F_n(x) \,|$$

where $F_n(x)$ is the empirical distribution function of the sample and $F(x;b,a)$ is the fitted distribution.

The modified Anderson-Darling A^2 statistic is given by

$$A^2 = -\frac{\sum_{i=1}^{n} (2i - 1)\left[\ln U_{(i)} + \ln\left(1 - U_{(n+1-i)}\right)\right]}{n} - n$$

where $U_{(i)} = F(X_{(i)};b,a)$ and $X_{(i)}$ is the ith-order statistic.

Unlike the R_w^2 statistic, neither D nor A^2 is affected by the choice of scale. Thus, we can use either X or $\ln X$ in the calculations.

Two-Parameter Weibull Distributions

Simulation Results

The results of David and Johnson (1948) imply that the distributions of D and A^2 do not depend upon the values of b and a. Similarly, the distribution of R_{we}^2 does not depend upon the values of b and a. These and some other invariance properties of the statistics are shown in the Appendix. Therefore, without loss of generality, the distributions of D, A^2, and R_{we}^2 can be obtained assuming $b = 1$ and $a = 3.6$.

To calculate the critical values of the statistics, we used IMSL (1979) to generate a fixed sample $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ of n order statistics from a uniform distribution. Then we transformed to the Weibull order statistics,

$$X_{(i)} = b\,[-\ln(1 - U_{(i)})]^{1/a}$$

where $b = 1$ and $a = 3.6$. The sample $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ was used to calculate maximum likelihood estimates of b and a. Finally we calculated D, A^2, and R_{we}^2 for the sample of size n. Values of n = 10, 1.5, . . . , 50; n = 60, 70, . . . , 100; n = 120, 140, . . . , 200; and n = 240, 280, . . . , 400 were used. Concern over the apparent variability of results for the Anderson-Darling critical values led us to try several preliminary simulations with 1,000 to 10,000 replications. Results confirmed that the type of variability seen in Table 2 could occur from one simulation to the next. Because we were interested in smoothing out irregularities found in other tables, we generated 50,000 values for each goodness-of-fit statistic. These 50,000 values were ranked, and the 80th, 85th, 90th, 95th, and 99th percentiles were determined. The values in Table 3 are the results of this simulation. For the Kolmogorov-Smirnov D statistic, as in Chandra and others (1981), we again multiplied the critical value by $n^{1/2}$. By repeating the simulation with a different set of 50,000 replications for various sample sizes, we determined that extending the simulation to 50,000 replications reduces the variability in the Anderson-Darling A^2 statistic critical values from the second to the third decimal point. This is important for $n > 50$ because the critical values become much closer together.

Critical Value Approximations

In an effort to smooth the critical values in Table 3 and simplify future use of the test statistics, we modeled the critical values as functions of sample size. The modeling resulted in the following separate equations for the critical values of the Kolmogorov-Smirnov D statistic at the 0.10, 0.05, and 0.01 levels of significance:

$$D_{0.10} = 0.82645983 - (0.199103/n^{1/2})$$

$$D_{0.05} = 0.89820336 - (0.221577/n^{1/2})$$

$$D_{0.01} = 1.04550210 - (0.282595/n^{1/2})$$

These curves fit the tabled values reasonably well, as shown in Figure 1. We made no attempt to force these curves to give the asymptotic values of Chandra and others (1981). Those asymptotic values were estimated by an extrapolation procedure described in their paper. Since the critical values in Table 3 are slightly above the estimated asymptotic values for large samples, we decided to fit the data of Table 3 without forcing the curve through Chandra's estimate.

Stephens' (1977) modification of the asymptotic critical values of the Anderson-Darling A^2 statistic for use with finite samples requires multiplying the value found by the test statistic by $(1 + 0.2/n^{1/2})$. The resulting value can then be compared to the asymptotic critical values shown in Table 2. Results of this procedure are plotted in Figure 2. Here the critical values in Table 3 are converted and compared to the asymptotic values. Figure 2 shows that the approximation is very good.

Modeling the critical values of the correlation statistic R^2_{we} produces the following separate equations for the 0.10, 0.05, and 0.01 levels of significance:

$$R_{0.10} = 0.99550280 - (3.46422/n) + (61.17125245/n^2) - (706.629/n^3) + (3047.57446/n^4)$$

$$R_{0.05} = 0.99373844 - (4.69737/n) + (91.36608058/n^2) - (1093.48/n^3) + (4804.52152/n^4)$$

$$R_{0.01} = 0.98826584 - (8.82798/n) + (205.65876975/n^2) - (2548.8/n^3) + (11329.68065/n^4).$$

These curves are plotted with the critical values in Figure 3.

Power Studies

To evaluate our results, we conducted a power study of the three statistics, D, A^2, and R^2_{we}, using four different distributions:

1. Uniform distribution on 0 to 1

2. Truncated normal with a mean of 1.4 and standard deviation of 0.35 (The distribution is truncated so that no values less than 0.00001 are allowed.)

3. Lognormal distribution where In X has a mean of 1.6 and standard deviation of 0.4

4. Gamma distribution with shape parameter equal to 2 and scale parameter equal to 1

The procedure involved generating 5,000 pseudorandom samples of size n from each of the four alternative distributions considered. We then calculated each of the three test statistics and compared them to their critical values from Table 3 and the

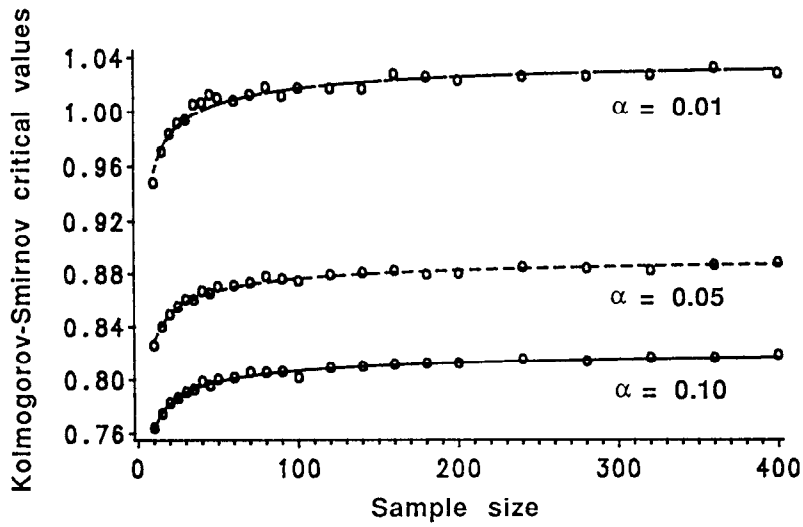

Figure 1 — Two-parameter Weibull Kolmogorov-Smirnov D statistic critical values with fitted model.

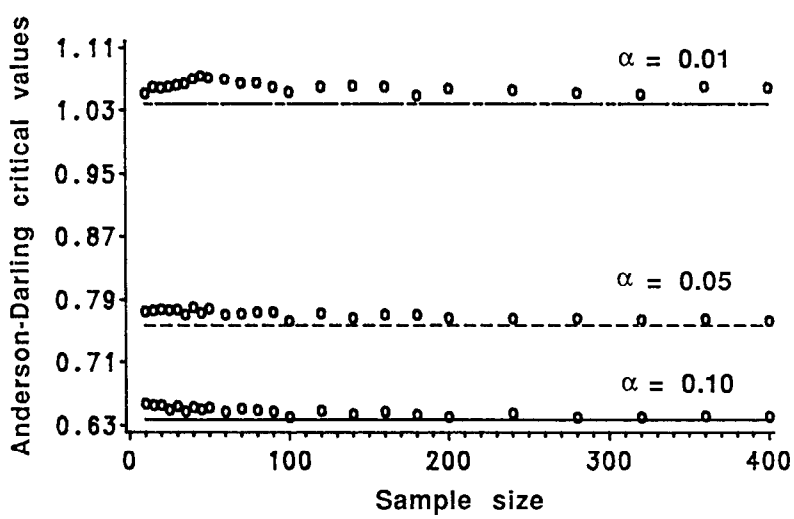

Figure 2 — Two-parameter Weibull Anderson-Darling A^2 statistic critical values with Stephens asymptotic model.

approximations to the critical values obtained from the equations for D, A^2, and R^2_{we}. In each case, we counted the number of rejections of the null hypothesis. We repeated this procedure for sample sizes of n = 20, 50, 80, 100, and 200.

Results of the power study are presented in Tables 4 to 6, which show that the Anderson-Darling A^2 statistic is generally superior to (that is, has a larger power than) both the Kolmogorov-Smirnov D and the correlation R^2_{we} statistics for the alternatives presented. Neither D nor R^2_{we} appears to be very powerful across this group of alternative distributions. One other feature illustrated in Tables 4 to 6 is the close agreement in power between the actual critical values and the values from the approximation formulas, which attests to the accuracy of the approximations.

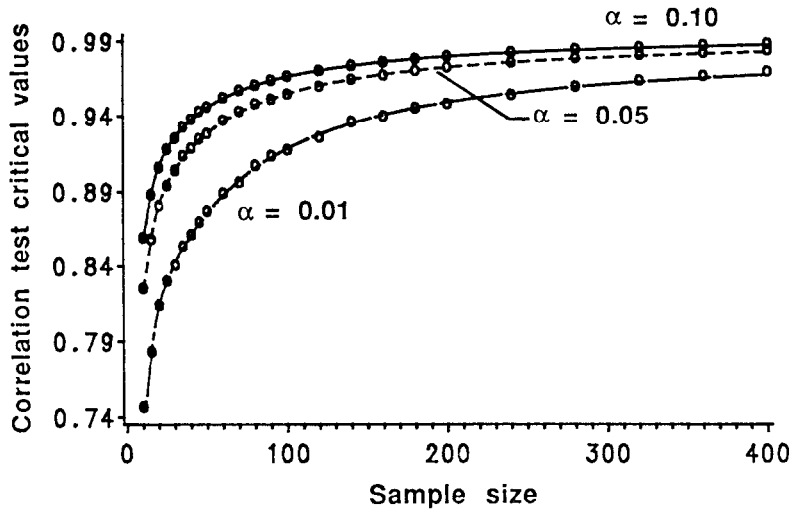

Figure 3 — Two-parameter Weibull correlation test R^2_{wc} statistic critical values with fitted model.

Three-Parameter Weibull Distributions

Simulation Results

Developing critical values for the three-parameter Weibull distribution goodness-of-fit critical statistics is more difficult because they depend upon the unknown shape parameter. Selecting the scale on which to perform the tests is also a problem. In the two-parameter case, using In $X_{(i)}$ instead of $X_{(i)}$ made the correlation statistic independent of the shape parameter. In the three-parameter case, the choice of which scale to use is not obvious. For the Kolmogorov-Smirnov D and the Anderson-Darling A^2 statistics, the two scales produce the same results. Therefore, we investigated D, A^2, R^2_w, and R^2_{we} using a = 2.0, 2.8, 3.6, 4.4, and 5.2. We chose b = 1.0 and c = 2.0. Invariance properties derived in the Appendix show that the critical values of the statistics depend only upon the shape parameter.

To calculate the critical values of the statistics, IMSL (1979) routines were again used to generate a fixed sample $U_{(1)}, U_{(2)}, \ldots, U_{(n)}$ of n order statistics from a uniform distribution. Then we transformed to the Weibull order statistics,

$$X_{(i)} = b[-\ln(1 - U_{(i)})]^{1/a} + c$$

for the different combinations of a, b, and c. Maximum likelihood estimates of a, b, and c were calculated for the sample $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$. Finally we calculated D, A^2, R^2_w, and R^2_{wc} for the sample of size n. Values of n = 20, 40, . . . , 200 were used. Because of the increased computer time required to calculate the maximum likelihood estimates for the three-parameter Weibull distribution compared with the two-parameter case, it was possible to do only 10,000 replications. Thus, for each sample size and choice of a, b, and c, 10,000 data sets were created and the goodness-of-fit statistics evaluated. For each statistic, the 10,000 values were ranked and the 80th, 85th, 90th, 95th and 99th percentiles were determined (Tables 4 to 10). Again the results for the Kolmogorov-Smirnov statistic have been multiplied by $n^{1/2}$.

Critical Value Approximations

We wanted to smooth the results and develop formulas that would approximate the critical values. Because the critical values vary slightly for different shape parameters, two models were tried for each statistic. One model ignores the differences in values due to shape and just expresses the general trend due to sample size. For this simple model, we determined that for the Kolmogorov-Smirnov D statistic and the Anderson-Darling A^2 statistic, the two-parameter models could be modified by subtracting a constant from the two-parameter model for the critical value. For the correlation statistics, producing an entirely new model was better. A second model, more complex, attempts to model the apparent quadratic nature of the effect of shape on the critical values. This refinement was added to the simple model developed for each of the three statistics. Thus, two models were created to predict the critical values of the Kolmogorov-Smirnov, Anderson-Darling, and correlation (original and extreme value scales) statistics.

Kolmogorov-Smirnov Models –

1. Two-parameter model plus a shift

$$K_{0.10} = 0.82645983 - (0.199103/n^{1/2}) - 0.04699184$$

$$K_{0.05} = 0.89820336 - (0.221577/n^{1/2}) - 0.05393332$$

$$K_{0.01} = 1.04550210 - (0.282595/n^{1/2}) - 0.06876439$$

2. Two-parameter model plus a shift and an adjustment for the estimated shape parameter values (labeled SHAPE in the formulas)

$$\begin{aligned} KS_{0.10} = {} & 0.82645983 - (0.199103/n^{1/2}) - 0.04699184 \\ & + 0.05302253 - [0.025511(\text{SHAPE})] + [0.002725911(\text{SHAPE})(\text{SHAPE})] \end{aligned}$$

$$\begin{aligned} KS_{0.05} = {} & 0.89820336 - (0.221577/n^{1/2}) - 0.05393332 \\ & + 0.05968916 - [0.0285431(\text{SHAPE})] + [0.00302431(\text{SHAPE})(\text{SHAPE})] \end{aligned}$$

$$\begin{aligned} KS_{0.01} = {} & 1.04550210 - (0.282595/n^{1/2}) - 0.06876439 \\ & + 0.0897581 - [0.0443486(\text{SHAPE})] + [0.004908494(\text{SHAPE})(\text{SHAPE})] \end{aligned}$$

Testing is done by multiplying the Kolmogorov-Smirnov D statistic by $n^{1/2}$ and comparing the result to the critical values from the formulas. We reject the hypothesis that a three-parameter Weibull fits the data if $n^{1/2}(D)$ is larger than the critical value.

Anderson-Darling Models –

1. Two-parameter model plus a shift

$$AD_{0.10} = 0.637 - 0.08561220$$

$$AD_{0.05} = 0.757 - 0.10568530$$

$$AD_{0.01} = 1.038 - 0.15088989$$

2. Two-parameter model plus a shift and an adjustment for shape

$$\mathrm{ADS}_{0.10} = 0.637 - 0.08561220 + 0.11048439 - [0.0535362(\mathrm{SHAPE})] + [0.005775685(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

$$\mathrm{ADS}_{0.05} = 0.757 - 0.10568530 + 0.12038387 - [0.0575924(\mathrm{SHAPE})] + [0.006105963(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

$$\mathrm{ADS}_{0.01} = 1.038 - 0.15088989 + 0.18351844 - [0.0880853(\mathrm{SHAPE})] + [0.009381216(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

Testing is done by multiplying the Anderson-Darling A^2 statistic by $(1.0 + 0.2/n^{1/2})$ and comparing to the critical values from the formulas. We reject the hypothesis that a three-parameter Weibull fits the data if $A^2(1 + 0.2/n^{1/2})$ is larger than the critical value.

Correlation Models – Original Scale –

1. New models

$$\mathrm{RW}_{0.10} = 0.99885011 - (1.45389/n) + (7.25250378/n^2)$$

$$\mathrm{RW}_{0.05} = 0.99877755 - (1.77495/n) + (8.67291150/n^2)$$

$$\mathrm{RW}_{0.01} = 0.99910494 - (2.66292/n) + (12.86169089/n^2)$$

2. New models plus an adjustment for shape

$$\mathrm{RWS}_{0.10} = 0.99885011 - (1.45389/n) + (7.25250378/n^2) - 0.0154459 + [0.008294643(\mathrm{SHAPE})] - [0.00101228(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

$$\mathrm{RWS}_{0.05} = 0.99877755 - (1.77495/n) + (8.67291150/n^2) - 0.0215137 + [0.01150321(\mathrm{SHAPE})] - [0.00139732(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

$$\mathrm{RWS}_{0.01} = 0.99910494 - (2.66292/n) + (12.86169089/n^2) - 0.0403293 + [0.02112679(\mathrm{SHAPE})] - [0.00250893(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

Testing is done by comparing R_w^2 calculated using $X_{(i)}$ to the critical values from the formulas. We reject the hypothesis that a three-parameter Weibull fits the data if R_w^2 is less than the critical value.

Correlation Models-Extreme Value Scale –

1. New models

$$\mathrm{RWE}_{0.10} = 0.99411418 - (1.81407/n) + (12.38547217/n^2)$$

$$\mathrm{RWE}_{0.05} = 0.99229032 - (2.24194/n) + (16.33414042/n^2)$$

$$\mathrm{RWE}_{0.01} = 0.98757887 - (3.37283/n) + (26.99680370/n^2)$$

2. New models plus an adjustment for shape

$$\mathrm{RWES}_{0.10} = 0.99411418 - (1.81407/n) + (12.38547217/n^2) - 0.00705129 + [0.003971786(\mathrm{SHAPE})] - [0.000508929(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

$$\mathrm{RWES}_{0.05} = 0.99229032 - (2.24194/n) + (16.33414042/n^2) - 0.00551925 + [0.00348(\mathrm{SHAPE})] - [0.000492187(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

$$\mathrm{RWES}_{0.01} = 0.98757887 - (3.37283/n) + (26.99680370/n^2) + 0.001807429 + [0.0006810714(\mathrm{SHAPE})] - [0.000299107(\mathrm{SHAPE})(\mathrm{SHAPE})]$$

Testing is done by comparing R^2_{we} calculated using ln $X_{(i)}$ to the critical values from the formulas. We reject the hypothesis that a three-parameter Weibull fits the data if R^2_{we} is less than the critical value.

Power Studies

We compared the four statistics and their approximations for critical values in a power study using two Weibull distributions and two alternative distributions:

1. Weibull distribution with $a = 2.8$, $b = 1.0$, and $c = 2.0$

2. Weibull distribution with $a = 4.4$, $b = 1.0$, and $c = 2.0$

3. Lognormal distribution where ln X has a mean of 1.6 and standard deviation of 4, and 2.0 added to each observation

4. Gamma distribution with shape equal to 2, and 2.0 added to each observation

We chose Weibull distributions for use in the three-parameter case to evaluate the difference between the simple and complex models for critical values. This was not necessary in the two-parameter case, because the statistics did not depend on the true shape parameter.

The procedure involved generating 2,000 pseudorandom samples of size n from each of the four distributions considered. We then calculated each of the four test statistics and compared them with the critical values obtained from the models presented in the previous section. In each case we counted the number of rejections of the null hypothesis. This procedure was repeated for n = 20, 50, 80, 100, and 200.

Results of the power study are presented in Tables 11 to 13, which show that nothing was gained by using the more complex critical value formulas. The approximation models for critical values do produce the correct percentage of rejections for the two Weibull distributions, indicating the adequacy of the models. For the non-Weibull distributions, the correlation statistics R^2_{w} and R^2_{we} are the most powerful of the test statistics. The Anderson-Darling A^2 statistic is slightly more powerful than the Kolmogorov-Smirnov D statistic.

Comparing the two-parameter results with the three-parameter results shows that for a given sample size, the Anderson-Darling statistic is much more powerful in the

two-parameter case than in the three-parameter case. Conversely, the correlation test is more powerful in the three-parameter case than in the two-parameter case.

Conclusions

Four general conclusions are readily apparent from the results of this study:

1. Of the three statistics considered, the Anderson-Darling A^2 statistic appears to be the statistic of choice for testing the goodness-of-fit of a two-parameter Weibull distribution to a set of data because of the larger power it displayed in the power studies. The choice might be different if censored data were used since the correlation statistic has the potential advantage of being easily modified for type I and type II censored observations.

2. For testing the three-parameter Weibull goodness-of-fit, the correlation statistics R^2_{we} and R^2_{w} appear to be the best because of their greater power against the alternative distributions considered.

3. The critical value approximations appear to be very good for the range of sample sizes considered. For the two-parameter Weibull distribution critical values, this is seen in the close agreement of the plotted curves and the critical values as shown in Figures 1 to 3. For the three-parameter Weibull distribution critical values, the quality of the approximations can be seen in the power study where powers for the Weibull distributions tested were near the significance level of the test.

4. The power of the tests is very dependent upon the true distributional form of the data. For example, in the two-parameter Weibull distribution power study, power against the uniform distribution was much higher than against the truncated normal.

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Table 1 –Two-parameter Weibull critical values for the Kolmogorov-Smirnov D statistic based on maximum likelihood estimates

Sample size	Critical values at various significance levels								
	Littell and others (1979)[a]					Chandra and others (1981)			
	0.20	0.15	0.10	0.05	0.01	0.10	0.05	0.025	0.01
10	0.689	0.721	0.759	0.822	0.949	0.760	0.819	0.880	0.944
15	0.701	0.736	0.778	0.840	0.972	–	–	–	–
20	0.707	0.738	0.783	0.854	0.984	0.779	0.843	0.907	0.973
25	0.710	0.745	0.785	0.850	0.975	–	–	–	–
30	0.712	0.745	0.789	0.854	0.980	–	–	–	–
35	0.716	0.745	0.793	0.864	1.023	–	–	–	–
40	0.715	0.746	0.791	0.860	0.999	–	–	–	–
50	–	–	–	–	–	0.790	0.856	0.922	0.988
∞[b]	–	–	–	–	–	0.803	0.874	0.939	1.007

[a] Values multiplied by $n^{1/2}$ to convert to the scale of Chandra and others (1981)
[b] These are theoretical asymptotic values given by Chandra and others (1981).

Table 2–Two-parameter Weibull critical values for the Anderson-Darling A^2 statistic based on maximum likelihood estimates

Sample size	Critical values at various significance levels								
	Wozniak and Warren (1984)			Stephens (1977)			Littell and others (1979)		
	0.10	0.05	0.01	0.10	0.05	0.01	0.10	0.05	0.01
10	0.614	0.724	0.966	0.599	0.712	0.976	0.616	0.730	0.988
15	0.618	0.734	0.984	0.606	0.720	0.987	0.622	0.740	0.994
20	0.623	0.740	1.005	0.610	0.725	0.994	0.625	0.744	1.012
25	0.626	0.745	1.021	0.612	0.728	0.998	0.625	0.745	1.019
30	0.629	0.747	1.025	0.615	0.730	1.001	0.626	0.741	1.007
35	0.625	0.745	1.026	0.616	0.732	1.004	0.626	0.739	1.026
40	0.628	0.748	1.030	0.617	0.734	1.006	0.627	0.739	1.001
50	0.635	0.753	1.034	0.619	0.736	1.009	–	–	–
∞[a]	–	–	–	0.637	0.757	1.038	–	–	–

[a] Theoretical asymptotic values.

Table 3–Two-parameter Weibull critical values for D, A^2, and R^2_{we}

Sample size	Statistic	Critical values at various significance levels				
		0.20	0.15	0.10	0.05	0.01
10	D	0.6894	0.7210	0.7637	0.8254	0.9477
	A^2	0.5032	0.5513	0.6171	0.7277	0.9876
	R^2_{we}	0.8924	0.8785	0.8589	0.8246	0.7462
15	D	0.6991	0.7308	0.7742	0.8393	0.9702
	A^2	0.5060	0.5538	0.6222	0.7372	1.0065
	R^2_{we}	0.9148	0.9038	0.8875	0.8579	0.7826
20	D	0.7053	0.7379	0.7822	0.8488	0.9830
	A^2	0.5064	0.5557	0.6265	0.7433	1.0120
	R^2_{we}	0.9287	0.9194	0.9055	0.8803	0.8133
25	D	0.7095	0.7440	0.7860	0.8545	0.9910
	A^2	0.5061	0.5550	0.6236	0.7458	1.0179
	R^2_{we}	0.9381	0.9295	0.9175	0.8936	0.8296
30	D	0.7126	0.7460	0.7904	0.8599	0.9936
	A^2	0.5109	0.5602	0.6302	0.7489	1.0235
	R^2_{we}	0.9441	0.9363	0.9249	0.9035	0.8403
35	D	0.7147	0.7484	0.7922	0.8596	1.0051
	A^2	0.5060	0.5552	0.6255	0.7448	1.0279
	R^2_{we}	0.9500	0.9431	0.9324	0.9133	0.8531
40	D	0.7191	0.7533	0.7982	0.8665	1.0062
	A^2	0.5111	0.5610	0.6320	0.7550	1.0362
	R^2_{we}	0.9538	0.9472	0.9374	0.9182	0.8609
45	D	0.7191	0.7520	0.7949	0.8647	1.0123
	A^2	0.5104	0.5608	0.6303	0.7495	1.0414
	R^2_{we}	0.9577	0.9518	0.9426	0.9244	0.8692
50	D	0.7212	0.7559	0.7997	0.8697	1.0097
	A^2	0.5117	0.5618	0.6336	0.7559	1.0405
	R^2_{we}	0.9604	0.9545	0.9456	0.9280	0.8766
60	D	0.7211	0.7560	0.8009	0.8706	1.0078
	A^2	0.5107	0.5606	0.6299	0.7506	1.0414
	R^2_{we}	0.9652	0.9600	0.9519	0.9367	0.8884
70	D	0.7262	0.7597	0.8049	0.8726	1.0124
	A^2	0.5133	0.5635	0.6350	0.7529	1.0384
	R^2_{we}	0.9686	0.9639	0.9567	0.9426	0.8956
80	D	0.7263	0.7612	0.8050	0.8774	1.0179
	A^2	0.5124	0.5633	0.6343	0.7560	1.0404
	R^2_{we}	0.9711	0.9668	0.9602	0.9477	0.9067
90	D	0.7267	0.7618	0.8054	0.8756	1.0113
	A^2	0.5117	0.5622	0.6331	0.7571	1.0361
	R^2_{we}	0.9733	0.9694	0.9634	0.9508	0.9131
100	D	0.7250	0.7570	0.8010	0.8740	1.0170
	A^2	0.5071	0.5573	0.6269	0.7467	1.0309
	R^2_{we}	0.9756	0.9720	0.9662	0.9543	0.9169
120	D	0.7296	0.7635	0.8084	0.8785	1.0166
	A^2	0.5109	0.5626	0.6359	0.7575	1.0392
	R^2_{we}	0.9785	0.9752	0.9701	0.9596	0.9254
140	D	0.7312	0.7644	0.8093	0.8803	1.0164
	A^2	0.5120	0.5624	0.6323	0.7529	1.0416
	R^2_{we}	0.9806	0.9778	0.9733	0.9641	0.9356
160	D	0.7324	0.7665	0.8108	0.8816	1.0271
	A^2	0.5123	0.5636	0.6360	0.7580	1.0419
	R^2_{we}	0.9824	0.9797	0.9756	0.9669	0.9393
180	D	0.7312	0.7661	0.8117	0.8788	1.0250
	A^2	0.5106	0.5618	0.6328	0.7581	1.0320
	R^2_{we}	0.9839	0.9815	0.9777	0.9699	0.9448
200	D	0.7326	0.7665	0.8118	0.8796	1.0225
	A^2	0.5096	0.5587	0.6309	0.7550	1.0411
	R^2_{we}	0.9851	0.9829	0.9795	0.9722	0.9476

Table 3–Two-parameter Weibull critical values for D, A^2, and R^2_{we} –con.

Sample size	Statistic	Critical values at various significance levels				
		0.20	0.15	0.10	0.05	0.01
240	D	0.7343	0.7699	0.8149	0.8846	1.0256
	A^2	0.5112	0.5637	0.6364	0.7550	1.0408
	R^2_{we}	0.9870	0.9851	0.9821	0.9754	0.9537
280	D	0.7346	0.7697	0.8132	0.8835	1.0257
	A^2	0.5111	0.5617	0.6316	0.7557	1.0380
	R^2_{we}	0.9885	0.9867	0.9841	0.9783	0.9593
320	D	0.7352	0.7692	0.8157	0.8819	1.0268
	A^2	0.5095	0.5624	0.6325	0.7545	1.0372
	R^2_{we}	0.9895	0.9879	0.9854	0.9803	0.9633
360	D	0.7381	0.7722	0.8159	0.8861	1.0322
	A^2	0.5131	0.5644	0.6341	0.7562	1.0479
	R^2_{we}	0.9904	0.9889	0.9866	0.9819	0.9663
400	D	0.7380	0.7720	0.8180	0.8880	1.0280
	A^2	0.5127	0.5636	0.6338	0.7545	1.0469
	R^2_{we}	0.9912	0.9899	0.9878	0.9835	0.9694

Table 4–Simulated powers of two-parameter Weibull test statistics[a] -α = 0.10

Distribution	Sample size	D		A^2		R^2_{we}	
		T	F	T	F	T	F
Uniform	20	0.3480	0.3506	0.4932	0.5132	0.2944	0.2944
	50	0.7006	0.7024	0.8968	0.9048	0.5214	0.5250
	80	0.8780	0.8776	0.9864	0.9870	0.6988	0.7038
	100	0.9444	0.9400	0.9970	0.9970	0.7966	0.7930
	200	0.9992	0.9992	1.0000	1.0000	0.9832	0.9796
Truncated normal	20	0.1220	0.1250	0.1196	0.1340	0.1232	0.1232
	50	0.1358	0.1368	0.1484	0.1590	0.1432	0.1452
	80	0.1524	0.1520	0.1816	0.1902	0.1650	0.1670
	100	0.1670	0.1602	0.2098	0.2106	0.1620	0.1616
	200	0.2308	0.2272	0.3182	0.3192	0.2104	0.1976
Lognormal	20	0.2562	0.2582	0.3130	0.3292	0.3036	0.3036
	50	0.4970	0.4992	0.6656	0.6790	0.5796	0.5832
	80	0.6898	0.6892	0.8560	0.8642	0.7814	0.7856
	100	0.7868	0.7750	0.9312	0.9318	0.8820	0.8794
	200	0.9720	0.9710	0.9976	0.9976	0.9970	0.9962
Gamma	20	0.1256	0.1274	0.1134	0.1248	0.1038	0.1038
	50	0.1394	0.1404	0.1472	0.1572	0.0928	0.0956
	80	0.1618	0.1612	0.1962	0.2040	0.1150	0.1166
	100	0.1814	0.1684	0.2250	0.2272	0.1092	0.1078
	200	0.2642	0.2612	0.3468	0.3492	0.1690	0.1496

[a] Values from table (T) and formula (F).

Table 5–Simulated powers of two-parameter Weibull test statistics[a] –α = 0.05

Distribution	Sample size	*D*		A^2		R^2_{wc}	
		T	F			T	F
Uniform	20	0.2444	0.2462	0.3696	0.3886	0.1940	0.1936
	50	0.5596	0.5642	0.8212	0.8350	0.3392	0.3472
	80	0.7798	0.7822	0.9660	0.9720	0.4984	0.4936
	100	0.8864	0.8800	0.9912	0.9914	0.5868	0.5888
	200	0.9984	0.9978	1.0000	1.0000	0.9080	0.8830
Truncated normal	20	0.0610	0.0622	0.0590	0.0674	0.0612	0.0610
	50	0.0722	0.0734	0.0884	0.0958	0.0700	0.0718
	80	0.0822	0.0838	0.1130	0.1214	0.0910	0.0894
	100	0.0934	0.0894	0.1290	0.1312	0.0928	0.0928
	200	0.1442	0.1380	0.2132	0.2190	0.1228	0.1118
Lognormal	20	0.1518	0.1538	0.2164	0.2322	0.1754	0.1746
	50	0.3608	0.3632	0.5464	0.5640	0.3694	0.3754
	80	0.5466	0.5494	0.7796	0.7886	0.5998	0.5912
	100	0.6608	0.6512	0.8794	0.8820	0.7194	0.7202
	200	0.9360	0.9320	0.9960	0.9962	0.9806	0.9738
Gamma	20	0.0650	0.0654	0.0612	0.0668	0.0486	0.0482
	50	0.0754	0.0760	0.0862	0.0940	0.0364	0.0384
	80	0.0862	0.0874	0.1176	0.1262	0.0452	0.0442
	100	0.0972	0.0934	0.1332	0.1364	0.0380	0.0386
	200	0.1678	0.1590	0.2424	0.2492	0.0566	0.0466

[a] Values from table (T) and formula (F)

Table 6–Simulated powers of two-parameter Weibull test statistics[a] – α = 0.01

Distribution	Sample size	*D*		A^2		R^2_{wc}	
		T	F	T	F	T	F
Uniform	20	0.0900	0.1002	0.1672	0.1776	0.0620	0.0610
	50	0.2976	0.3090	0.5930	0.6186	0.1052	0.1030
	80	0.5384	0.5430	0.8708	0.8808	0.1562	0.1460
	100	0.6798	0.6744	0.9538	0.9572	0.1820	0.1850
	200	0.9748	0.9718	1.0000	1.0000	0.3906	0.3732
Truncated normal	20	0.0124	0.0150	0.0148	0.0162	0.0146	0.0138
	50	0.0164	0.0178	0.0238	0.0284	0.0212	0.0212
	80	0.0230	0.0238	0.0334	0.0368	0.0276	0.0274
	100	0.0234	0.0230	0.0462	0.0480	0.0322	0.0322
	200	0.0460	0.0420	0.0862	0.0910	0.0426	0.0414
Lognormal	20	0.0484	0.0550	0.0884	0.0928	0.0416	0.0404
	50	0.1594	0.1666	0.3252	0.3452	0.0544	0.0572
	80	0.2950	0.2998	0.5824	0.6014	0.1292	0.1206
	100	0.3952	0.3900	0.7154	0.7212	0.1726	0.1772
	200	0.7862	0.7720	0.9778	0.9792	0.6340	0.6102
Gamma	20	0.0148	0.0164	0.0172	0.0188	0.0056	0.0054
	50	0.0204	0.0218	0.0218	0.0250	0.0018	0.0018
	80	0.0238	0.0248	0.0344	0.0392	0.0032	0.0030
	100	0.0276	0.0270	0.0424	0.0460	0.0026	0.0026
	200	0.0508	0.0478	0.0908	0.0992	0.0024	0.0022

[a] Values from table (T) and formula (F).

Table 7–Three-parameter Weibull critical values for the Kolmogorov-Smirnov *D* statistic

Sample size	Shape	Critical values at various significance levels				
		0.20	0.15	0.10	0.05	0.01
20	2.0	0.6619	0.6909	0.7281	0.7844	0.8949
	2.8	0.6507	0.6798	0.7182	0.7723	0.8998
	3.6	0.6516	0.6807	0.7169	0.7764	0.8908
	4.4	0.6538	0.6816	0.7191	0.7755	0.8882
	5.2	0.6552	0.6856	0.7222	0.7799	0.9083
40	2.0	0.6925	0.7242	0.7640	0.8228	0.9645
	2.8	0.6774	0.7084	0.7457	0.8083	0.9284
	3.6	0.6691	0.6989	0.7419	0.8001	0.9234
	4.4	0.6704	0.7014	0.7425	0.7982	0.9228
	5.2	0.6723	0.7039	0.7444	0.8020	0.9215
60	2.0	0.6917	0.7242	0.7653	0.8288	0.9528
	2.8	0.6832	0.7150	0.7591	0.8234	0.9466
	3.6	0.6770	0.7088	0.7514	0.8149	0.9512
	4.4	0.6754	0.7049	0.7444	0.8040	0.9295
	5.2	0.6747	0.7057	0.7444	0.8102	0.9233
80	2.0	0.6968	0.7290	0.7683	0.8336	0.9776
	2.8	0.6905	0.7227	0.7585	0.8238	0.9472
	3.6	0.6842	0.7129	0.7495	0.8139	0.9347
	4.4	0.6816	0.7120	0.7504	0.8095	0.9239
	5.2	0.6798	0.7129	0.7522	0.8130	0.9374
100	2.0	0.7010	0.7320	0.7730	0.8380	0.9630
	2.8	0.6940	0.7250	0.7660	0.8300	0.9530
	3.6	0.6870	0.7180	0.7570	0.8180	0.9450
	4.4	0.6870	0.7160	0.7560	0.8170	0.9320
	5.2	0.6860	0.7170	0.7540	0.8170	0.9430
120	2.0	0.7055	0.7339	0.7778	0.8413	0.9760
	2.8	0.6956	0.7285	0.7690	0.8358	0.9552
	3.6	0.6923	0.7219	0.7624	0.8238	0.9508
	4.4	0.6847	0.7164	0.7548	0.8128	0.9355
	5.2	0.6868	0.7164	0.7580	0.8216	0.9508
140	2.0	0.7076	0.7371	0.7797	0.8484	0.9809
	2.8	0.6969	0.7265	0.7655	0.8306	0.9525
	3.6	0.6898	0.7218	0.7620	0.8235	0.9407
	4.4	0.6910	0.7206	0.7584	0.8212	0.9347
	5.2	0.6863	0.7158	0.7549	0.8152	0.9383
160	2.0	0.7096	0.7425	0.7842	0.8513	0.9854
	2.8	0.6982	0.7286	0.7678	0.8310	0.9651
	3.6	0.6944	0.7248	0.7627	0.8285	0.9550
	4.4	0.6919	0.7197	0.7589	0.8273	0.9537
	5.2	0.6868	0.7185	0.7564	0.8184	0.9449
180	2.0	0.7084	0.7406	0.7795	0.8439	0.9901
	2.8	0.7003	0.7299	0.7728	0.8345	0.9606
	3.6	0.6950	0.7258	0.7661	0.8278	0.9526
	4.4	0.6936	0.7245	0.7661	0.8305	0.9552
	5.2	0.6923	0.7205	0.7594	0.8211	0.9485
200	2.0	0.7085	0.7410	0.7835	0.8499	0.9786
	2.8	0.6986	0.7311	0.7722	0.8358	0.9617
	3.6	0.6958	0.7269	0.7651	0.8287	0.9645
	4.4	0.6901	0.7212	0.7637	0.8273	0.9631
	5.2	0.6901	0.7227	0.7623	0.8245	0.9532

Table 8–Three-parameter Weibull critical values for the Anderson-Darling A^2 statistic

Sample size	Shape	Critical values at various significance levels				
		0.20	0.15	0.10	0.05	0.01
20	2.0	0.4296	0.4677	0.5191	0.6044	0.7977
	2.8	0.4190	0.4539	0.5016	0.5825	0.7828
	3.6	0.4183	0.4536	0.5038	0.5892	0.8000
	4.4	0.4164	0.4541	0.5092	0.5978	0.7909
	5.2	0.4202	0.4592	0.5106	0.5946	0.7939
40	2.0	0.4587	0.5022	0.5640	0.6646	0.9112
	2.8	0.4343	0.4735	0.5319	0.6295	0.8476
	3.6	0.4236	0.4661	0.5196	0.6226	0.8527
	4.4	0.4239	0.4637	0.5237	0.6159	0.8402
	5.2	0.4296	0.4693	0.5283	0.6252	0.8361
60	2.0	0.4579	0.5011	0.5592	0.6588	0.9152
	2.8	0.4417	0.4824	0.5434	0.6445	0.8628
	3.6	0.4343	0.4760	0.5278	0.6295	0.8682
	4.4	0.4273	0.4657	0.5154	0.6106	0.8505
	5.2	0.4250	0.4654	0.5260	0.6226	0.8471
80	2.0	0.4588	0.5031	0.5654	0.6636	0.9168
	2.8	0.4414	0.4810	0.5430	0.6370	0.8801
	3.6	0.4330	0.4739	0.5283	0.6319	0.8665
	4.4	0.4311	0.4700	0.5237	0.6196	0.8295
	5.2	0.4295	0.4715	0.5295	0.6274	0.8535
100	2.0	0.4653	0.5082	0.5742	0.6760	0.9271
	2.8	0.4500	0.4918	0.5515	0.6486	0.8832
	3.6	0.4363	0.4787	0.5377	0.6326	0.8664
	4.4	0.4290	0.4697	0.5300	0.6243	0.8463
	5.2	0.4304	0.4716	0.5290	0.6185	0.8445
120	2.0	0.4678	0.5109	0.5750	0.6770	0.9133
	2.8	0.4499	0.4923	0.5490	0.6512	0.8759
	3.6	0.4404	0.4835	0.5431	0.6422	0.8627
	4.4	0.4293	0.4708	0.5269	0.6253	0.8420
	5.2	0.4325	0.4740	0.5327	0.6337	0.8664
140	2.0	0.4686	0.5136	0.5754	0.6790	0.9357
	2.8	0.4471	0.4886	0.5487	0.6540	0.8770
	3.6	0.4440	0.4868	0.5425	0.6399	0.8685
	4.4	0.4345	0.4730	0.5320	0.6243	0.8384
	5.2	0.4294	0.4707	0.5237	0.6193	0.8427
160	2.0	0.4674	0.5156	0.5824	0.6912	0.9610
	2.8	0.4525	0.4948	0.5512	0.6503	0.8754
	3.6	0.4414	0.4829	0.5395	0.6419	0.8740
	4.4	0.4385	0.4766	0.5314	0.6355	0.8728
	5.2	0.4321	0.4726	0.5302	0.6272	0.8457
180	2.0	0.4663	0.5098	0.5760	0.6821	0.9390
	2.8	0.4522	0.4927	0.5533	0.6541	0.8750
	3.6	0.4435	0.4856	0.5408	0.6403	0.8891
	4.4	0.4439	0.4881	0.5482	0.6512	0.9052
	5.2	0.4332	0.4693	0.5299	0.6320	0.8587
200	2.0	0.4668	0.5115	0.5706	0.6742	0.9376
	2.8	0.4537	0.5019	0.5613	0.6664	0.9276
	3.6	0.4431	0.4844	0.5414	0.6359	0.8772
	4.4	0.4359	0.4774	0.5360	0.6293	0.8506
	5.2	0.4321	0.4722	0.5318	0.6280	0.8706

Table 9–Three-parameter Weibull critical values for the correlation test R_w^2 statistic

Sample size	Shave	Critical values at various significance levels				
		0.20	0.15	0.10	0.05	0.01
20	2.0	0.9527	0.9471	0.9395	0.9252	0.8873
	2.8	0.9571	0.9522	0.9457	0.9334	0.8993
	3.6	0.9574	0.9527	0.9459	0.9336	0.9029
	4.4	0.9565	0.9520	0.9456	0.9340	0.9016
	5.2	0.9565	0.9519	0.9447	0.9323	0.8996
40	2.0	0.9707	0.9672	0.9622	0.9527	0.9250
	2.8	0.9744	0.9715	0.9674	0.9606	0.9414
	3.6	0.9755	0.9728	0.9693	0.9628	0.9448
	4.4	0.9753	0.9726	0.9689	0.9625	0.9456
	5.2	0.9747	0.9715	0.9675	0.9609	0.9448
60	2.0	0.9790	0.9765	0.9729	0.9659	0.9479
	2.8	0.9818	0.9797	0.9767	0.9717	0.9585
	3.6	0.9825	0.9808	0.9782	0.9739	0.9618
	4.4	0.9825	0.9808	0.9784	0.9741	0.9639
	5.2	0.9819	0.9800	0.9773	0.9725	0.9607
80	2.0	0.9832	0.9812	0.9783	0.9728	0.9581
	2.8	0.9858	0.9843	0.9821	0.9784	0.9693
	3.6	0.9864	0.9849	0.9832	0.9798	0.9708
	4.4	0.9863	0.9848	0.9829	0.9796	0.9720
	5.2	0.9858	0.9843	0.9823	0.9787	0.9702
100	2.0	0.9859	0.9843	0.9820	0.9778	0.9654
	2.8	0.9881	0.9869	0.9853	0.9821	0.9729
	3.6	0.9887	0.9875	0.9860	0.9832	0.9769
	4.4	0.9887	0.9876	0.9860	0.9835	0.9767
	5.2	0.9884	0.9872	0.9856	0.9827	0.9761
120	2.0	0.9879	0.9865	0.9844	0.9809	0.9704
	2.8	0.9900	0.9889	0.9875	0.9847	0.9778
	3.6	0.9903	0.9894	0.9881	0.9858	0.9803
	4.4	0.9904	0.9894	0.9881	0.9858	0.9811
	5.2	0.9899	0.9889	0.9876	0.9851	0.9795
140	2.0	0.9893	0.9881	0.9863	0.9830	0.9727
	2.8	0.9913	0.9903	0.9891	0.9869	0.9809
	3.6	0.9916	0.9907	0.9896	0.9877	0.9829
	4.4	0.9916	0.9908	0.9897	0.9878	0.9837
	5.2	0.9913	0.9905	0.9895	0.9875	0.9827
160	2.0	0.9904	0.9893	0.9877	0.9849	0.9769
	2.8	0.9922	0.9914	0.9902	0.9881	0.9830
	3.6	0.9925	0.9918	0.9909	0.9892	0.9851
	4.4	0.9925	0.9918	0.9909	0.9892	0.9848
	5.2	0.9922	0.9915	0.9905	0.9887	0.9847
180	2.0	0.9915	0.9905	0.9890	0.9866	0.9797
	2.8	0.9929	0.9922	0.9912	0.9894	0.9849
	3.6	0.9933	0.9927	0.9918	0.9903	0.9868
	4.4	0.9932	0.9925	0.9916	0.9901	0.9859
	5.2	0.9931	0.9924	0.9915	0.9899	0.9863
200	2.0	0.9921	0.9913	0.9900	0.9877	0.9810
	2.8	0.9936	0.9929	0.9919	0.9901	0.9861
	3.6	0.9939	0.9933	0.9926	0.9912	0.9881
	4.4	0.9939	0.9934	0.9925	0.9912	0.9880
	5.2	0.9937	0.9931	0.9923	0.9907	0.9877

Table 10–Three-parameter Weibull critical values for the correlation test R^2_{we} statistic

Sample size	Shape	Critical values at various significance levels				
		0.20	0.15	0.10	0.05	0.01
20	2.0	0.9407	0.9351	0.9271	0.9134	0.8817
	2.8	0.9475	0.9423	0.9347	0.9231	0.8902
	3.6	0.9497	0.9446	0.9373	0.9240	0.8912
	4.4	0.9492	0.9437	0.9361	0.9219	0.8839
	5.2	0.9501	0.9448	0.9363	0.9222	0.8842
40	2.0	0.9633	0.9595	0.9538	0.9444	0.9205
	2.8	0.9667	0.9629	0.9572	0.9474	0.9239
	3.6	0.9678	0.9639	0.9588	0.9493	0.9223
	4.4	0.9676	0.9637	0.9581	0.9480	0.9204
	5.2	0.9669	0.9629	0.9570	0.9469	0.9201
60	2.0	0.9726	0.9699	0.9661	0.9595	0.9404
	2.8	0.9746	0.9716	0.9670	0.9582	0.9381
	3.6	0.9754	0.9724	0.9681	0.9605	0.9383
	4.4	0.9757	0.9728	0.9682	0.9598	0.9383
	5.2	0.9747	0.9716	0.9670	0.9581	0.9364
80	2.0	0.9779	0.9756	0.9724	0.9662	0.9517
	2.8	0.9791	0.9768	0.9733	0.9668	0.9519
	3.6	0.9795	0.9771	0.9733	0.9666	0.9498
	4.4	0.9794	0.9767	0.9731	0.9662	0.9462
	5.2	0.9793	0.9767	0.9725	0.9651	0.9462
100	2.0	0.9813	0.9792	0.9765	0.9715	0.9584
	2.8	0.9821	0.9801	0.9771	0.9714	0.9562
	3.6	0.9827	0.9807	0.9775	0.9719	0.9547
	4.4	0.9825	0.9803	0.9774	0.9711	0.9550
	5.2	0.9822	0.9800	0.9765	0.9703	0.9526
120	2.0	0.9836	0.9819	0.9793	0.9747	0.9634
	2.8	0.9844	0.9826	0.9799	0.9753	0.9631
	3.6	0.9844	0.9825	0.9797	0.9739	0.9602
	4.4	0.9844	0.9825	0.9796	0.9744	0.9598
	5.2	0.9842	0.9821	0.9791	0.9735	0.9567
140	2.0	0.9853	0.9838	0.9816	0.9777	0.9684
	2.8	0.9863	0.9846	0.9820	0.9782	0.9664
	3.6	0.9860	0.9842	0.9814	0.9768	0.9635
	4.4	0.9862	0.9844	0.9817	0.9767	0.9628
	5.2	0.9861	0.9843	0.9815	0.9763	0.9630
160	2.0	0.9867	0.9852	0.9832	0.9795	0.9704
	2.8	0.9873	0.9859	0.9837	0.9794	0.9698
	3.6	0.9874	0.9858	0.9835	0.9789	0.9680
	4.4	0.9873	0.9856	0.9831	0.9782	0.9667
	5.2	0.9871	0.9856	0.9829	0.9780	0.9650
180	2.0	0.9879	0.9867	0.9849	0.9815	0.9731
	2.8	0.9883	0.9869	0.9849	0.9815	0.9725
	3.6	0.9883	0.9868	0.9849	0.9806	0.9697
	4.4	0.9880	0.9864	0.9842	0.9797	0.9674
	5.2	0.9884	0.9868	0.9845	0.9801	0.9681
200	2.0	0.9891	0.9880	0.9864	0.9831	0.9756
	2.8	0.9892	0.9879	0.9858	0.9822	0.9730
	3.6	0.9893	0.9878	0.9858	0.9822	0.9731
	4.4	0.9893	0.9879	0.9859	0.9818	0.9704
	5.2	0.9891	0.9876	0.9854	0.9813	0.9695

Table 11 – Simulated powers of three-parameter Weibull test statistics[a] – α = 0.10

Distribution	Simple size	D		A^2		R_w^2		R_{wc}^2	
		S	C	S	C	S	C	S	C
Weibull	20	0.1135	0.0965	0.0925	0.0745	0.1200	0.0950	0.1325	0.1220
(a = 2.8)	50	0.1055	0.0980	0.0945	0.0865	0.0820	0.0765	0.0965	0.0935
	80	0.1065	0.1030	0.1090	0.1010	0.0940	0.0835	0.1015	0.1000
	100	0.1095	0.1040	0.1035	0.0955	0.0880	0.0755	0.1045	0.1035
	200	0.1095	0.1050	0.1125	0.1080	0.0910	0.0745	0.0790	0.0780
Weibull	20	0.1000	0.0710	0.0960	0.0650	0.1080	0.0675	0.1115	0.0835
	50	0.0840	0.0750	0.0795	0.0740	0.0675	0.0580	0.0860	0.0810
	80	0.0960	0.0940	0.0875	0.0860	0.0760	0.0795	0.1005	0.0960
	100	0.0960	0.0975	0.0790	0.0810	0.0680	0.0730	0.0960	0.0985
	200	0.1010	0.1085	0.0915	0.1005	0.0640	0.1195	0.1010	0.1060
Lognormal	20	0.1480	0.1245	0.1000	0.0805	0.2420	0.1985	0.4550	0.4400
	50	0.1880	0.1720	0.2265	0.1985	0.3790	0.3025	0.4350	0.4015
	80	0.2425	0.2165	0.3250	0.2935	0.4995	0.4020	0.5530	0.5095
	100	0.2875	0.2590	0.3975	0.3595	0.5850	0.4760	0.6235	0.5725
	200	0.5135	0.4895	0.6750	0.6435	0.7975	0.6555	0.7855	0.7305
Gamma	20	0.1745	0.1395	0.0980	0.0735	0.2560	0.2150	0.6030	0.5895
	50	0.1465	0.1155	0.1650	0.1230	0.3320	0.2420	0.4650	0.4075
	80	0.1640	0.1285	0.1825	0.1395	0.3865	0.2585	0.4635	0.3970
	100	0.1515	0.1255	0.1800	0.1410	0.4065	0.2470	0.4580	0.3645
	200	0.2080	0.1805	0.2580	0.2210	0.5425	0.2760	0.4675	0.3410

[a] Values from simple (S) or complex (C) formula.

Table 12 – Simulated powers of three-parameter Weibull test statistics[a] – α = 0.05

Distribution	Simple size	D		A^2		R_w^2		R_{wc}^2	
		S	C	S	C	S	C	S	C
Weibull	20	0.0660	0.0495	0.0500	0.0365	0.0680	0.0510	0.0830	0.0775
(a = 2.8)	50	0.0555	0.0515	0.0470	0.0455	0.0460	0.0370	0.0440	0.0435
	80	0.0530	0.0500	0.0590	0.0545	0.0470	0.0435	0.0590	0.0595
	100	0.0535	0.0500	0.0520	0.0480	0.0425	0.0365	0.0500	0.0515
	200	0.0500	0.0470	0.0535	0.0495	0.0450	0.0340	0.0390	0.0405
Weibull	20	0.0540	0.0345	0.0550	0.0355	0.0575	0.0380	0.0620	0.0485
(a = 4.4)	50	0.0455	0.0405	0.0400	0.0370	0.0295	0.0260	0.0395	0.0385
	80	0.0465	0.0460	0.0425	0.0395	0.0325	0.0355	0.0490	0.0480
	100	0.0475	0.0490	0.0425	0.0395	0.0310	0.0360	0.0570	0.0560
	200	0.0525	0.0555	0.0425	0.0465	0.0260	0.0585	0.0490	0.0490
Lognormal	20	0.0900	0.0705	0.0595	0.0490	0.1620	0.1275	0.3805	0.3775
	50	0.1175	0.0985	0.1535	0.1325	0.2865	0.2205	0.2975	0.2840
	80	0.1570	0.1410	0.2360	0.2065	0.4080	0.3050	0.4175	0.3945
	100	0.2000	0.1765	0.3040	0.2740	0.5025	0.3720	0.4855	0.4660
	200	0.4080	0.3805	0.5940	0.5680	0.7395	0.5525	0.6710	0.6440
Gamma	20	0.1045	0.0750	0.0570	0.0435	0.1765	0.1295	0.5325	0.5265
	50	0.0745	0.0585	0.0915	0.0705	0.2480	0.1530	0.3130	0.2800
	80	0.0870	0.0590	0.1070	0.0795	0.2985	0.1625	0.3025	0.2585
	100	0.0870	0.0645	0.1095	0.0835	0.3165	0.1705	0.2780	0.2335
	200	0.1285	0.1040	0.1740	0.1430	0.4275	0.1865	0.2815	0.2260

[a] Values from simple (S) or complex (C) formula

Table 13–Simulated powers of three-parameter Weibull test statistics[a] – α = 0.01

Distribution	Simple size	D		A^2		R_w^2		R_{wc}^2	
		S	C	S	C	S	C	S	C
Weibull (a = 2.8)	20	0.0160	0.0110	0.0115	0.0075	0.0190	0.0160	0.0280	0.0285
	50	0.0080	0.0070	0.0110	0.0085	0.0080	0.0040	0.0065	0.0065
	80	0.0120	0.0120	0.0110	0.0095	0.0095	0.0070	0.0110	0.0125
	100	0.0080	0.0060	0.0080	0.0070	0.0075	0.0045	0.0090	0.0090
	200	0.0070	0.0065	0.0085	0.0075	0.0075	0.0040	0.0065	0.0085
Weibull (a = 4.4)	20	0.0170	0.0110	0.0130	0.0090	0.0190	0.0150	0.0180	0.0165
	50	0.0080	0.0055	0.0065	0.0050	0.0050	0.0050	0.0060	0.0060
	80	0.0110	0.0115	0.0085	0.0090	0.0035	0.0055	0.0145	0.0120
	100	0.0075	0.0075	0.0090	0.0090	0.0060	0.0070	0.0185	0.0165
	200	0.0085	0.0100	0.0105	0.0110	0.0050	0.0165	0.0085	0.0080
Lognormal	20	0.0300	0.0210	0.0200	0.0145	0.0645	0.0415	0.2815	0.2860
	50	0.0330	0.0250	0.0620	0.0490	0.1610	0.1035	0.1205	0.1265
	80	0.0590	0.0460	0.1105	0.0865	0.2540	0.1470	0.1750	0.1895
	100	0.0705	0.0525	0.1500	0.1270	0.3315	0.1895	0.2185	0.2505
	200	0.2105	0.1815	0.4170	0.3795	0.5960	0.3370	0.4040	0.4590
Gamma	20	0.0270	0.0160	0.0185	0.0130	0.0680	0.0410	0.3960	0.4055
	50	0.0210	0.0145	0.0270	0.0170	0.1085	0.0550	0.1135	0.1230
	80	0.0165	0.0110	0.0285	0.0150	0.1460	0.0615	0.0740	0.0845
	100	0.0200	0.0135	0.0350	0.0220	0.1720	0.0655	0.0635	0.0785
	200	0.0415	0.0275	0.0705	0.0490	0.2795	0.0715	0.0575	0.0860

[a] Values from simple (S) or complex (C) formula.

Appendix – Some Invariance Properties of the Statistics

In this Appendix, we verify a number of invariance considerations that pertain to this study. We present results related to the scale choice, null distributions, and power.

Invariance of D and A^2 Under a Change to the Log Scale

Two-Parameter Case- In the two-parameter case, the estimate of the population distribution function is

$$F(x;\ \hat{b},\ \hat{a}) = 1 - \exp[-(x/\hat{b})^{\hat{a}}]$$

where $\hat{b}$ and $\hat{a}$ are the maximum likelihood estimates of the scale and shape parameters, respectively. Under a change to the log scale, the estimate becomes

$$P(\ln X \le y \mid \hat{a},\ \hat{b}) = \hat{F}_{\ln X}(y) = 1 - \exp[-e^{(y - \ln \hat{b})/(1/\hat{a})}]$$

That is, the uniform variables are estimated using

$$\hat{F}_{\ln X}(\ln x \mid \hat{a},\ \hat{b}) = 1 - \exp[-e^{(\ln x - \ln \hat{b})/(1/\hat{a})}] = 1 - \exp[-e^{\hat{a}\ \ln(x/b)}]$$

$$= 1 - \exp[-(x/\hat{b})^{\hat{a}}] = \hat{F}(x;\ \hat{b},\ \hat{a}) \qquad \text{(A.1)}$$

The estimated uniform variables are the same whichever scale is used. Equation (A. 1) holds even if the underlying population is not Weibull.

A similar conclusion holds under any given monotonic transformation.

Three-Parameter Case-In the three-parameter case, the estimate of the uniform variables takes the form

$$F(x;\ \hat{a},\ \hat{b},\ \hat{c}) = 1 - \exp\{-[(x - \hat{c})/\hat{b}]^{\hat{a}}\}$$

On the log scale, we estimate

$$\hat{F}_{\ln(X-\hat{c})}(y;\ \hat{a},\ \hat{b},\ \hat{c}) = 1 - \exp[-e^{(y - \ln \hat{b})/(1/\hat{a})}]$$

Consequently,

$$\hat{F}_{\ln(X-\hat{c})}[\ln(x - \hat{c});\ \hat{a},\ \hat{b},\ \hat{c}] = 1 - \exp[-e^{[\ln(x-\hat{c}) - \ln \hat{b}]/(1/\hat{a})}]$$

$$= 1 - \exp[-e^{\hat{a}\ \ln[(x - \hat{c})/\hat{b}]}]$$

$$= 1 - \exp[-(x - \hat{c})/\hat{b})^{\hat{a}}] = \hat{F}(x;\ \hat{a},\ \hat{b},\ \hat{c}) \qquad \text{(A.2)}$$

The estimated uniform variables are the same whichever scale is used. Because the uniform variables have the same values, A^2 and D are the same under both the original and log scales.

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Invariance When the Underlying Distribution Is Weibull

Two-Parameter Case – On the log scale

$$\frac{\log X_i - \log \hat{b}}{1/\hat{a}} \qquad (i = 1, 2, \ldots, n) \tag{A.3}$$

is distributed free of the location parameter log b and scale parameter a^{-1} (David and Johnson 1948; Lawless 1982, p. 147). From Equation (A.l) the distribution of estimated uniform observations does not depend on a and b, and hence the A^2 and D statistics do not depend on a and b.

The result in Equation (A.3) depends on a property of the maximum likelihood estimator (see Lawless (1982), p. 147) that

$$1/\hat{a}(a_0 \log X_1 + b_0, a_0 \log X_2 + b_0, \ldots, a_0 \log X_n + b_0)$$

$$= a_0/\hat{a}\,(\log X_1, \log X_2, \ldots, \log X_n)$$

$$\log \hat{b}(a_0 \log X_1 + b_0, a_0 \log X_2 + b_0, \ldots, a_0 \log X_n + b_0)$$

$$= a_0 \log \hat{b}\,(\log X_1, \log X_2, \ldots, \log X_n) + \log b_0 \tag{A.4}$$

That is, if X is transformed to $b_0\ X^{a_0}$, the maximum likelihood estimates of location and scale for the extreme value distribution obey Equation (A.4).

The R_{we} statistic, on the log scale, has a distribution that is free of the parameters since

$$\frac{\log X_{(i)} - \overline{\log X}}{\sqrt{\sum_{i=1}^{n} (\log X_{(i)} - \overline{\log X})^2}} = \frac{a[\log(X_{(i)}/b) - \overline{\log (X/b)}]}{a\sqrt{\sum_{i=1}^{n} [\log (X_{(i)}/b) - \overline{\log (X/b)})]^2}} \tag{A.5}$$

is a function of the standard exponential variables $Z_i = (X_{(i)}/b)^a$ and R_w is the correlation of these variables with the scores log $m_{w,i}$ where the $1/\hat{a}$ factor can be ignored in the calculation of correlation.

Three-Parameter Case-For the three-parameter case, Lemon (1975) gives the pivotal functions

$$\hat{a}_{abc} \overset{D}{=} \hat{a}_{a10} \qquad \frac{\hat{b}_{abc}}{b} \overset{D}{=} \hat{b}_{a10} \qquad \frac{\hat{c}_{abc} - c}{\hat{b}_{abc}} \overset{D}{=} \frac{\hat{c}_{a10}}{\hat{b}_{a10}}$$

where the equality in distribution is for the values of (a,b,c) indicated in the subscript. In other words, these quantities depend only on the underlying population shape parameter. Because

$$\frac{X_i - \hat{c}}{\hat{b}} = \frac{(X_i - c)/b}{\hat{b}/b} = \frac{(c - \hat{c})}{\hat{b}} \tag{A.6}$$

where the distribution of each term depends only on a, the estimated uniform variables in Equation (A.2) and hence the distributions of A^2 and D depend only on a, not the other parameters.

In the case of the R_w statistic, on the original scale,

$$\frac{X_{(i)} - \overline{X}}{\sqrt{\sum_{i=1}^{n} (X_{(i)} - \overline{X})^2}} = \frac{(X_i - c)/b - (\overline{X} - c)/b}{\sqrt{\sum_{i=1}^{n} [(X_i - c)/b - (\overline{X} - c)/b]^2}}$$

which is a function of the standard Weibull with parameters a_{10}. Since R_w is the correlation of these variables with the scores $m_{w,i}$ and the scores only depend on $\hat{a}$,, and hence a, the distribution of R_w depends only on a.

For the log scale,

$$\frac{\log(X_{(i)} - \hat{c}) - \overline{\log(X - \hat{c})}}{\sqrt{\sum_{i=1}^{n} [\log(X_{(i)} - \hat{c}) - \overline{\log(X - \hat{c})}]^2}} = \frac{\log[X_{(i)} - \hat{c})/b) - \overline{\log[(X - \hat{c})/b]}}{\sqrt{\sum_{i=1}^{n} \{\log[X_{(i)} - \hat{c})/b] - \overline{\log[(X - \hat{c})/b]}\}^2}} \quad \text{(A.7)}$$

Similar to Equation (A.6),

$$\log[(X_{(i)} - \hat{c})/b] = \log[(X_{(i)} - c)/b + (\hat{b}/b)(c - \hat{c})/\hat{b}]$$

has a distribution that depends only on a. Since R_w is the correlation of the variables (A.7) with the scores ln $m_{w,i}$, its distribution also depends only on a.

Invariance and Power

Because of the properties (Eq. (A.4)) of the maximum likelihood estimators, the power calculations pertain to a wider class of alternatives than is immediately apparent.

Two-Parameter Case – Let X be distributed as $G_O(x)$ and let $Y = b_0 X^{a_0}$. Then log Y_i = a,, log X_i + log b_0. On the log scale, $R_{w\ e}$ is a correlation and hence it has the same value whether calculated in terms of the log Y_i or the log X_i.

Further,

$$\frac{\text{g } Y_i - \log \hat{b}(y)}{1/\hat{a}(y)} = \frac{\log X_i - \log \hat{b}(x)}{1/\hat{a}(x)} \quad \text{(A.}$$

by the properties in Equation (A.4). Consequently, the estimated uniform variables are the same on both scales, so A^2 and D do not depend on the scale.

Thus, the power of each of the three tests remains the same for any lognormal distribution. Also, the powers for the uniform (0, 1) alternative hold for the uniform (0, θ), $\theta > 0$ and, more generally, for

$$G(x \mid \theta) = (x/\theta)^p \qquad (0 < x < \theta)$$

The basic relations between the maximum likelihood estimators on the two scales can be shown directly. Let $Y = b_0 X^{a_0}$. Then

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\begin{aligned}
L(y \mid a, b) &= \prod_{i=1}^{n} \frac{a}{b}\left(\frac{y_i}{b}\right)^{a-1} \exp\left[-\left(\frac{y_i}{b}\right)^{a}\right] \\
&= \prod_{i=1}^{n} \frac{a}{b}\left(\frac{b_0 x_i^{a_0}}{b}\right)^{a-1} \exp\left[-\left(\frac{b_0 x_i^{a_0}}{b}\right)^{a}\right] \\
&= \prod_{i=1}^{n} \frac{a}{b}\left(\frac{x_i}{(b/b_0)^{1/a_0}}\right)^{a_0 a - a_0} \exp\left[-\left(\frac{x_i}{(b/b_0)^{1/a_0}}\right)^{a_0 a}\right] \\
&= \prod_{i=1}^{n} \frac{a}{b^{1/a_0} b_0^{-1/a_0}} \left(\frac{x_i^{1-a_0}}{b_0}\right)\left(\frac{x_i}{(b/b_0)^{1/a_0}}\right)^{a_0 a - 1} \exp\left[-\left(\frac{z x_i}{(b/b_0)^{1/a_0}}\right)^{a_0 a}\right] \\
&= \left(\prod_{i=1}^{n} X_i\right)^{1-a_0} \frac{1}{b_0^n a_0^n} L[X \mid a a_0, (b/b_0)^{a_0}]
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Consequently, the maximum likelihood estimates based on x and y (or equivalently on $\ln y$) satisfy

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Because of this relation the estimated uniform variables are also equal.

$$\left(\frac{y_i}{\hat{b}(y)}\right)^{\hat{a}(y)} = \left(\frac{b_0 x_i^{a_0}}{b_0 \hat{b}^{a_0}(x)}\right)^{\hat{a}(x)/a_0} = \left(\frac{x_i}{\hat{b}(x)}\right)^{\hat{a}(x)} \qquad (i = 1, \ldots, n)$$

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