Bending of a Circular Sandwich Plate by Load Applied Through an Insert

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ABSTRACT

An analytical solution is presented for the bending of a circular sandwich plate by a normal load applied through a rigid clamped insert. The facings are assumed to be isotropic, while the core is assumed to possess negligible in-plane stiffnesses and a single transverse modulus of rigidity. During bending, the two facings have identical curvatures and the core shear stress is constant throughout the thickness of the core. Explicit solutions are presented for both a clamped and a simply supported plate.

Detailed derivations are provided for the plate bending and shear deflections, the core shear stress, and the facing radial and circumferential bending stresses, corresponding to each outer edge condition. Data for normalized representations of critical values of these quantities are presented in a series of design curves.
BENDING OF A CIRCULAR SANDWICH PLATE
BY LOAD APPLIED THROUGH AN INSERT

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INTRODUCTION

This research extends the theory of Ericksen for symmetrical bending of a circular sandwich plate to the case of bending by a point load applied through a rigid, clamped circular insert (or hub) at the center of the plate. The analysis is intended to provide guidance toward the design of intersandwich panel connectors. Solutions are obtained for plates clamped and simply supported at their outer edges.

The system is pictured in figure 1, showing the exterior plate and its interior section. The facings of the plate are composed of isotropic materials, while the core material is assumed to possess negligible in-plane stiffnesses. Shear deflections of the facings and effects due to bending of the core are neglected. As is usually assumed with small deflection bending of rectangular sandwich plates, the shear stress in the core is taken to be constant throughout the core thickness. Moreover, the facings are assumed to have identical curvatures during bending.

The theory and basic equations herein generalize those of Ericksen by allowing the sandwich top and bottom facings to be composed of either the same or different materials which have the same Poisson’s ratio. Analytical solutions are obtained for plate bending and shear deflections, core shear stress, and facing bending stresses corresponding to each plate outer edge condition. Normalized values of these quantities are presented in a series of design curves.

Notation can be found on page 31.

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2 Maintained at Madison, Wis., in cooperation with the University of Wisconsin.
Figure 1.--Cross section of a sandwich plate with load applied through a rigid circular insert.
GENERAL THEORY

Expressions follow for the plate deflection and stresses within the facings and core of a circular sandwich plate during bending due to a normal load. These expressions generalize those of Ericksen who assumed that the sandwich top and bottom facings, while having different thicknesses, were composed of the same material.

Plate Deflection

It is convenient to write the plate deflection $w$ as the sum of two parts: a contribution $w_b$ due to bending of the facings, and a contribution $w_s$ due to shear in the core. In particular, following Ericksen,

$$w = w_b + w_s$$

(1)

For a concentrated load $P$ applied at the center of a circular insert, the bending deflection can be written as

$$w_b = - \frac{1}{D} \left[ \frac{P}{8\pi} \left( \frac{2}{E} - 1 \right) + \frac{Jr^2}{2} + B\theta + C \right]$$

(2)

where

$$D = D_F + D_m$$

(3)

with

$$D_F = \frac{E_1 t_1^3 + E_2 t_2^3}{12\lambda}$$

(4)

and

$$D_m = \frac{E_1 t_1 E_2 t_2 h^2}{(E_1 t_1 + E_2 t_2)\lambda}$$

(5)

where $t_1$ and $t_2$ are the thicknesses of the top and bottom facings, respectively, $h$ is the distance separating facing midplanes, and $E_1$ and $E_2$ are the values of Young's modulus for the top and bottom facing materials, respectively. Furthermore,

$$\lambda = 1 - \nu^2$$

(6)

where $\nu$ is the value of Poisson's ratio for both facing materials. The minus sign in equation (2) implies a downward deflection in the direction of the load $P$. The variable $r$ denotes radial position as measured from the center of the plate.

The shear deflection can be written as

$$w_s = \frac{D_m}{\partial w} F^2 \left[ \frac{P}{2} - \frac{1}{\alpha E} \frac{F}{\partial \phi} + \frac{E}{\alpha} \frac{g}{\partial \theta} + H \right]$$

(7)
where 
\[ \alpha^2 = \frac{hGD}{D_F D_m} \]  
(8)

with G being the core modulus of rigidity. The symbols \( I_0 \) and \( K_0 \) represent modified Bessel functions of the first and second kind, respectively, and \( \frac{\partial}{\partial r} \) of order zero. The constants \( J, B, C, L, F, \) and \( H \) in equations (2) and (7) are evaluated using the boundary conditions at the insert rim \( r = b \) and the plate outer edge \( r = a \).

### Core Shear Stress

The shear stress in the core is given by Ericksen3 as

\[ \tau = \frac{D_F \alpha^2}{h} \frac{d\omega_s}{dr} \]  
(9)

Utilizing equation (7), this shear stress becomes

\[ \tau = \frac{h}{2mh} \left\{ \frac{P}{2\pi} + \frac{\partial}{\partial \alpha r} \left[ I_1 (\alpha r) - H_1 (\alpha r) \right] \right\} \]  
(10)

### Facing Rending Stresses

Ignoring any variations in the thickness direction, the radial stress in the top facing can be written as the sum of two contributions: \(^4\)

\[ \sigma = \sigma_{rb} + \sigma_{rs} \]  
(11)

The contribution \( \sigma_{rb} \) due to bending, is given by

\[ \sigma_{rb} = -\frac{P}{2t_1} \left( \frac{d^2 \omega_b}{dr^2} + \frac{\partial}{\partial r} \frac{d\omega_b}{dr} \right) \]  
(12)

The contribution \( \sigma_{rs} \), due to shear in the core, is given by

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\(^4\)Stresses in the bottom facing are obtained from the same formulas by replacing top facing thickness \( t_1 \) with bottom facing thickness \( t_2 \) and reversing the signs of the stresses.
Combining equations (12) and (2) yields

\[
\sigma = \frac{D_F}{t_1 h} \left( \frac{d^2 \omega}{dr^2} + \frac{d\omega}{dr} \right) \tag{13}
\]

Combining equations (12) and (2) yields

\[
\sigma_{rb} = \frac{D_m}{t_1 h} \left\{ \frac{p}{4\pi} \left[ (1+\mu) 2\alpha r + \frac{1-\mu}{2} \right] + J(1+\mu) - \frac{B(1-\mu)^2}{r^2} \right\} \tag{14}
\]

Combining equations (3) and (7),

\[
\sigma_{rs} = \frac{D_m}{t_1 h} \left\{ -\frac{p}{2\pi} (1-\mu) + \alpha L \left[ \frac{L_1(xr)}{xr} - (1-\mu) \frac{K_1(xr)}{xr} \right] \right\}
\]

\[
+ \alpha \left[ K_0(xr) + (1-\mu) \frac{K_1(xr)}{xr} \right] \tag{15}
\]

In a similar fashion, the circumferential stress can be written as the sum of two contributions:

\[
\sigma_\theta = \sigma_{\theta b} + \sigma_{\theta s} \tag{16}
\]

The contribution of \( \sigma_{\theta b} \), due to bending, is given by

\[
\sigma_{\theta b} = -\frac{D_m}{t_1 h} \left( \frac{d^2 \omega}{dr^2} + \frac{d\omega}{dr} \right) \tag{17}
\]

The contribution \( \sigma_{\theta s} \), due to shear in the core, is given by

\[
\sigma_{\theta s} = \frac{D_m}{t_1 h} \left( \frac{d^2 \omega}{dr^2} + \frac{d\omega}{dr} \right) \tag{18}
\]

Combining equations (17) and (2) yields

\[
\sigma_{\theta b} = \frac{D_m}{t_1 h} \left\{ \frac{p}{4\pi} \left[ (1+\mu) 2\alpha r - \frac{1-\mu}{2} \right] + J(1+\mu) + \frac{B(1-\mu)^2}{r^2} \right\} \tag{19}
\]
Combining equations (18) and (7),

\[
\frac{\Gamma}{\beta} = -\frac{D_m}{D_{nl}} \left\{ \frac{P}{(1-\nu)} \left[ (1-\nu) \frac{(1+\nu)}{r} \right] + F \left[ (1-\nu)K_1(\nu r) \right] \right\} \]

**ANALYTICAL DEVELOPMENT**

The plate bending and shear deflections, the core shear stress, and the facing radial and circumferential bending stresses are completely determined once the six constants, \( J, B, C, L, F, \) and \( H \), are evaluated. These constants are obtained from the boundary conditions at the insert rim \( r = b \) and at the plate outer edge \( r = a \). Derivations follow for the cases of clamped and simply supported plates, both having an insert clamped at its rim.

**Clamped Outer Edge**

At the rim of a clamped insert, the radial slopes \( \frac{dw_b}{dr} \) and \( \frac{dw_s}{dr} \) must vanish. These slopes are obtained from equations (2) and (7) as

\[
\frac{dw_b}{dr} = -\frac{1}{\beta} \left[ \frac{rF}{r} (\nu \gamma \beta - \frac{1}{2}) + \nu F + \frac{B}{b} \right]
\]

Forcing these slopes to vanish at \( r = b \) yields

\[
\frac{F}{\beta} (\nu \gamma \beta - \frac{1}{2}) + bF + \frac{B}{b} = 0 \tag{21}
\]

\[
\frac{F}{2} + \frac{B}{b} = 0 \tag{22}
\]

If the plate outer edge \( r = a \) is clamped, the deflections, \( w_b \) and \( w_s \), as well as the radial slopes \( \frac{dw_b}{dr} \) and \( \frac{dw_s}{dr} \) must vanish there. Therefore,

\[
\frac{P}{8} \cdot a^2 (\nu \gamma - 1) + \frac{J a^2}{2} + B \cdot k^2 + C = 0 \tag{23}
\]

\[
\frac{P}{2} \cdot a^2 + \frac{F}{A} (\nu \gamma a^2) + \frac{F}{A} K_1(\nu \gamma a) + H = 0 \tag{24}
\]
Solving equations (21) through (26) for the six constants and utilizing the results in equations (2) and (7) yields

\[
\begin{align*}
\omega_b &= - \frac{P}{\pi^2 h^3} \left[ \frac{1}{4} \left( \frac{r^2}{a^2} - \frac{a^2}{2} \right) - \frac{1}{4} \left( \frac{r^2}{a^2} - \frac{a^2}{2} \right)^2 \left( \frac{a}{2} - \frac{b}{2}\right) + \frac{2}{4} \frac{a^2 b}{h} \right] r^2 \\
\omega_s &= - \frac{PD_m}{2 \pi^2 h^3 r^2} \left[ a \frac{bK_1'(ab) - aK_1'(ab)}{ab} - bK_1'(ab) \right] \\
\end{align*}
\]

The core shear stress \( \tau \) becomes

\[
\begin{align*}
\tau &= \frac{PD_m}{2 \pi^2 h^3} \left[ \frac{1}{4} \left( \frac{r^2}{a^2} - \frac{a^2}{2} \right) - \frac{1}{4} \left( \frac{r^2}{a^2} - \frac{a^2}{2} \right)^2 \left( \frac{a}{2} - \frac{b}{2}\right) + \frac{2}{4} \frac{a^2 b}{h} \right] r^2 \\
\end{align*}
\]

Proceeding further, the facing radial stresses become

\[
\begin{align*}
\sigma_r &= \frac{PD_m}{4 \pi^2 h^3 r^2} \left[ 1 + \left( \frac{r}{a} \right)^2 \right] \left( 1 + \frac{1}{a^2} \right) + \left( 1 + \frac{1}{a^2} \right)^2 \left[ \frac{b^2}{a^2} \right] \left( \frac{r}{a} \right)^2 \\
\sigma_r &= \frac{PD_m}{2 \pi^2 h^3 r^2} \left[ 1 - \left( \frac{r}{a} \right)^2 \right] + \frac{\pi r^2 K_0'(sr) + (1-\pi)K_1'(sr)}{\pi^2 a^2 r K_1'(sr)} \\
\end{align*}
\]
Finally, the facing circumferential stresses can be written as

\begin{equation}
\sigma^c_{0b} = \frac{PD_m}{4\pi J_1} \left[ \frac{\mu}{a} + \frac{(1+\mu)K_0}{a} + \frac{b^2}{a^2 - b^2} \left[ (1+\mu) - \frac{1 - \mu}{(r/a)^2} \right] \frac{K_0}{a} \right]
\end{equation}

\begin{equation}
\sigma^c_{0s} = \frac{PD_m}{2\pi J_1} \left[ \frac{\mu - 1}{(r/a)^2} - \frac{\mu \alpha r K_0}{\alpha^2 a K_1} - \frac{(1-\mu)K_1}{(r/a)^2} \right]
\end{equation}

where

\begin{equation}
\Phi = \frac{\mu \alpha r I_0 + (1-\mu)I_1}{\alpha^3 ab r} + \frac{I_1}{K_1} \left[ \frac{\mu \alpha r K_0}{\alpha^2 a K_1} - \frac{(1-\mu)K_1}{(r/a)^2} \right]
\end{equation}

Equations (27) through (34) present the complete deflection and stress states within the plate if its outer edge is clamped.

**Simply Supported Outer Edge**

If the plate outer edge is simply supported, the deflections \(w_b\) and \(w_s\) and the radial stresses \(\sigma^b_{rb}\) and \(\sigma^s_{rs}\) must vanish there. To insure that the radial stresses are zero at \(r = a\), utilize equations (14) and (15) to get

\begin{equation}
\frac{P}{2\pi a} \left[ (1+\mu) \frac{\alpha r}{a} + \frac{1 - \mu}{2} \right] + J(1+\mu) - \frac{b(1-\mu)}{a^2} = 0
\end{equation}

\begin{equation}
-\frac{P}{2\pi a} (1-\mu) + \alpha \left[ I_0 (\alpha a) - (1-\mu) \frac{I_1 (\alpha a)}{\alpha a} \right] + b \left[ K_0 (\alpha a) + (1-\mu) \frac{K_1 (\alpha a)}{\alpha a} \right] = 0
\end{equation}

Equations (35) and (36), along with equations (21) through (24), are six simultaneous equations for the unknown constants corresponding to the case of a simply supported outer edge. After evaluating these constants, the plate deflections become
\[ \omega_b = -\frac{Pa^2}{16\pi D} \left[ 2\left(1 - \frac{r}{a}\right)^2 + \frac{b^2}{a} + \frac{r}{a} \right] + 4\left(\frac{b}{a}\right)^2 \frac{r}{a} \left[ \frac{(1+\mu) \frac{b}{a} - 1}{(1-\mu) \left(\frac{b}{a}\right)^2 - (1+\mu)} \right] \]  

\[ + \left[ \frac{2(1+\mu) \frac{b}{a} - (1-\mu) \left(1 - \frac{b^2}{a}\right)}{(\mu-1) \left(\frac{b}{a}\right)^2 - (1+\mu)} \right] \left[ 1 - \left(\frac{r}{a}\right)^2 \right] \]  

where

\[ \omega_s = -\frac{PD_m}{2\pi D \alpha^2} \left[ \frac{b^2}{r} - \frac{r}{\alpha_{ab}^2} \right] \]  

\[ \nu = (\alpha a)^2 \left[ I_0(\alpha a) K_0(\alpha r) - I_1(\alpha r) K_0(\alpha a) \right] \]

\[ + (1-\mu) \left[ \alpha a I_1(\alpha b) - \alpha b I_1(\alpha a) \right] \left[ K_0(\alpha r) - K_0(\alpha a) \right] \]  

\[ + (\alpha b K_1(\alpha b) - \alpha a K_1(\alpha a)) \left[ I_0(\alpha r) - I_0(\alpha a) \right] \]  

and

\[ \zeta = (\alpha a) \left[ I_0(\alpha a) K_1(\alpha b) + I_1(\alpha b) K_0(\alpha a) \right] \]

\[ - (1-\mu) \left[ \alpha a I_1(\alpha b) - \alpha b I_1(\alpha a) \right] \left[ K_0(\alpha r) - K_0(\alpha a) \right] \]  

After considerable differentiation and simplification, the core shear stress can be written as

\[ \gamma = \frac{PD_m}{2\pi D h} \left[ \frac{1}{r} - \delta \right] \]  

where

\[ \delta = \frac{\alpha a}{b} \left[ I_0(\alpha a) K_1(\alpha r) + K_0(\alpha a) I_1(\alpha r) \right] \]

\[ + \left[ \frac{1-\mu}{b} \left[ \alpha a I_1(\alpha r) \right] \left[ K_1(\alpha a) - (\alpha b) K_1(\alpha b) \right] \right] \]

\[ - K_1(\alpha r) \left[ \alpha a I_1(\alpha a) - \alpha b I_1(\alpha b) \right] \]
Proceeding as with the clamped edge case, the radial stresses in the facings become

\[
\sigma_{rb} = \frac{P_{b} m}{4 \pi D_{1} h} \left\{ \frac{1 - \nu}{2} + (1 + \nu) \frac{b}{a} \right\} + \frac{1 - \nu}{(r/b)^{2}} \left\{ \frac{(1 + \nu) \frac{b}{a} - 1}{(1 + \nu) + (1 - \nu) \left( \frac{b}{a} \right)^{2}} + \frac{1 + \nu}{2} \right\} \frac{2(1 + \nu) \frac{b}{a} - (1 - \nu) \left( 1 - \frac{b^{2}}{a^{2}} \right)}{(1 + \nu) + (1 - \nu) \left( \frac{b}{a} \right)^{2}} \right\}
\]

\[
\sigma_{rs} = -\frac{P_{s} m}{2 \pi D_{1} h} \left\{ \frac{1 - \nu}{(\alpha r)^{2}} + \frac{1}{\alpha \beta \Gamma(\alpha \beta)} \right\}
\]

(41)

where

\[
\begin{align*}
\Omega &= \left[ 2 \pi I_{0}(\alpha r) + (1 - \nu) I_{1}(\alpha r) \right] \left\{ (1 - \mu) [\alpha K_{1}(\alpha b) - \alpha a K_{1}(\alpha a)] - \alpha a K_{0}(\alpha a) \right\} \\
&\quad + \left[ 2 \pi K_{0}(\alpha r) + (1 - \nu) K_{1}(\alpha r) \right] \left\{ (1 - \mu) [\alpha a I_{1}(\alpha b) - \alpha a I_{1}(\alpha a)] + (\alpha a)^{2} I_{0}(\alpha a) \right\} \\
&+ \left[ 2 \pi I_{0}(\alpha r) + (1 - \nu) I_{1}(\alpha r) \right] \left\{ (1 - \mu) [\alpha K_{1}(\alpha b) - \alpha a K_{1}(\alpha a)] - \alpha a K_{0}(\alpha a) \right\} \\
&\quad + \left[ 2 \pi K_{0}(\alpha r) + (1 - \nu) K_{1}(\alpha r) \right] \left\{ (1 - \mu) [\alpha a I_{1}(\alpha b) - \alpha a I_{1}(\alpha a)] + (\alpha a)^{2} I_{0}(\alpha a) \right\}
\end{align*}
\]

(41a)

Finally, the facing circumferential stresses take the forms

\[
\sigma_{0b} = \frac{P_{b} m}{4 \pi D_{1} h} \left\{ \frac{1 - \nu}{2} + (1 + \nu) \frac{b}{a} \right\} \frac{1 - (1 + \nu) \frac{b}{a} - (1 - \nu) \left( 1 - \frac{b^{2}}{a^{2}} \right)}{(1 + \nu) + (1 - \nu) \left( \frac{b}{a} \right)^{2}} \right\}
\]

\[
\sigma_{0s} = \frac{P_{s} m}{2 \pi D_{1} h} \left\{ \frac{1 - \nu}{\alpha \beta \Gamma(\alpha \beta)} \right\} \left\{ 2(1 + \nu) \frac{b}{a} - (1 - \nu) \left( 1 - \frac{b^{2}}{a^{2}} \right) \right\} + \frac{1 + \nu}{2} \frac{(1 + \nu) \frac{b}{a} - (1 - \nu) \left( 1 - \frac{b^{2}}{a^{2}} \right)}{(1 + \nu) + (1 - \nu) \left( \frac{b}{a} \right)^{2}} \right\}
\]

(42)

\[
\begin{align*}
\sigma_{0} &= \frac{P_{s} m}{2 \pi D_{1} h} \left\{ \frac{1 - \nu}{\alpha \beta \Gamma(\alpha \beta)} \right\} \left\{ \frac{P(1 - \mu) \frac{1}{\alpha r} + \nu a \left[ I_{0}(\alpha r) + \frac{1}{\alpha r} \right]}{\beta r} \right\} \\
&\quad + \frac{1 - \nu}{\alpha \beta \Gamma(\alpha \beta)} \left[ \frac{(1 - \mu) \frac{1}{\alpha r} + \nu a \left[ I_{0}(\alpha r) + \frac{1}{\alpha r} \right]}{\beta r} \right] \right\}
\end{align*}
\]

(43)

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Equations (37) through (45) present the complete deflection and stress states within the plate if the outer edge is simply supported.

**DESIGN PARAMETERS AND MAXIMUM STRESSES AND DEFLECTIONS**

The quantities of primary interest in design are the maximum values of the deflection of the plate, and the maximum values of the stresses within its core and facings. In particular, data are required for the maximum plate bending and shear deflections, facing radial and circumferential bending stresses, and core shear stress.

The solutions thus far presented are obviously not in forms suitable for design calculations. With this in mind, design parameters are next defined, and the maximum values of interest are written in terms of normalized design coefficients.

Consider first the plate bending deflection $w_b$. The maximum value of this deflection always occurs at the insert rim defined by $r = b$. Define the radius ratio

$$\gamma = \frac{b}{a} \quad (46)$$

Utilizing equation (46) and evaluating equations (27) and (37) at $r = b$, it becomes possible to write the resulting expressions into the form

$$w_b(b) = -\frac{Pb^2}{16} \frac{1}{D} 1 \quad (47)$$

where

$$k_1^{(CLMP)} = 1 - \gamma^2 - \frac{4\gamma^2(\omega \gamma)^2}{1 - \gamma^2} \quad (48)$$

$$\frac{1}{2\pi a} \left\{ \frac{1 - \frac{1}{2} \gamma_1 (ab) + \frac{1}{\alpha a} \left[ I_0 (\alpha a) - (1-u) \frac{1}{\alpha a} \right]}{I_0 (\alpha a) I_1 (ab)} \right\}$$

$$\frac{1 - \frac{1}{2} \gamma_1 (ab) + \frac{1}{\alpha a} \left[ I_0 (\alpha a) - (1-u) \frac{1}{\alpha a} \right]}{I_0 (\alpha a) I_1 (ab)} \quad (44)$$

and

$$r = \frac{P_2}{2\pi a}$$

$$\frac{1 - \frac{1}{2} \gamma_1 (ab) + \frac{1}{\alpha a} \left[ I_0 (\alpha a) - (1-u) \frac{1}{\alpha a} \right]}{I_0 (\alpha a) I_1 (ab)} \quad (45)$$

Equations (37) through (45) present the complete deflection and stress states within the plate if the outer edge is simply supported.
Equations (48) and (49) present normalized values of the maximum plate bending deflection corresponding to clamped and simply supported outer edges, respectively.

Next, consider the maximum plate shear deflection $w_s$, which likewise always occurs at the insert rim. Defining

$$A = a a$$

makes it possible to evolve equations (28) and (38) into the form

$$w_s(b) = -\frac{P D}{2 a D D_f a^2} k_2$$

where

$$k_2^{(CLMF)} = -\xi_Y$$

$$\frac{[\gamma I_1(\gamma A) - I_1(A)] [K_o(A) - K_o(\gamma A)] + [\gamma K_1(\gamma A) - K_1(A)] [I_0(A) - I_0(\gamma A)]}{\gamma A [I_1(A) K_1(\gamma A) - I_1(\gamma A) K_1(A)]}$$

and

$$k_2^{(S.S.)} = -\xi_Y = -\frac{\Gamma}{\gamma A}$$

in which

$$\Gamma = A [I_0(A) K_o(\gamma A) - I_0(\gamma A) K_o(A)] - (1-\mu) [\gamma I_1(\gamma A) - I_1(A)] [K_o(A) - K_o(\gamma A)]$$

$$+ [\gamma K_1(\gamma A) - K_1(A)] [I_0(A) - I_0(\gamma A)]$$
and

$$
\Delta = A\{I_0(A)K_1(\gamma A) + I_1(\gamma A)K_0(A)\} - (1-\mu)\{I_1(A)K_1(\gamma A) - I_1(\gamma A)K_1(A)\}
$$

Equations (52) and (53) present the normalized maximum plate shear deflection, for each edge support case, in terms of the radius ratio $\gamma$ and a "size-property" parameter $A$.

The location of the maximum core shear stress is not apparent—in general, it will occur at a radial location intermediate to the insert rim and plate outer edge. Writing $\beta = \alpha r$, differentiating the resulting expressions for $\sigma$ with respect to $\beta$, and setting the results equal to zero yields appropriate equations for the desired locations. In particular, the location of the maximum core shear stress within the clamped plate is given by the solution of

$$
\left[I_1(A) - \gamma I_1(\gamma A)\right]\left[K_o(\beta) + K_2(\beta)\right] - \left[\gamma K_1(\gamma A) - K_1(A)\right]\left[I_0(\beta) + I_2(\beta)\right] = \frac{2\gamma A}{\beta^2}\left[I_1(A)K_1(\gamma A) - I_1(\gamma A)K_1(A)\right]
$$

For the simply supported plate, the required location is given by the solution of

$$
\gamma A\left\{A\left[I_0(A)K_1(\gamma A) + I_1(\gamma A)K_0(A)\right] - (1-\mu)\left[I_1(A)K_1(\gamma A) - I_1(\gamma A)K_1(A)\right]\right\} = \beta^2\begin{bmatrix}K_o(\beta) + K_2(\beta)\end{bmatrix}\begin{bmatrix}1 - \frac{1}{2}\gamma I_1(\gamma A) - I_1(A)\end{bmatrix}
$$

Equations (29) and (39) for the core shear stress can each then be maximized and written in the form

$$
T_{\text{MAX}} = \frac{PD}{2\pi hD^3}
$$

where

$$
k_3^{(\text{CLMP})} = \frac{\gamma A}{\beta} - \frac{I_1(\beta)\left[\gamma K_1(\gamma A) - K_1(A)\right] - K_1(\beta)\left[\gamma I_1(\gamma A) - I_1(A)\right]}{I_1(A)K_1(\gamma A) - I_1(\gamma A)K_1(A)}
$$
with \( \beta \) given by the solution of equation (54), and

\[
\nu_A (\beta, \gamma) = \frac{\gamma A}{\beta} - \frac{\psi}{\beta}
\]

in which

\[
\psi = A [ I_0 (\alpha) K_1 (\beta) + I_1 (\beta) K_0 (\alpha)] - (1-\mu) I_1 (\beta) \left[ \gamma K_1 (\gamma A) - K_1 (A) \right]
\]

(58a)

\[\mathbf{k}_4 (S, S) = \frac{\pi d}{4 \pi d + h k_4}
\]

(59)

Equations (60) and (61) give normalized values of the maximum facing radial bending stresses for a clamped or simply supported outer edge, respectively.
As with the radial facing stress, only that portion of the circumferential facing stress due to bending is considered further. The maximum value of this circumferential stress does not necessarily occur at the insert rim but, in general, occurs at a radial location intermediate to the rim and the plate outer edge. However, because the maximum radial stress occurs at the insert rim and because, as will be shown by the numerical results, the difference between the maximum circumferential stress and its value at the insert rim is not large, it is appropriate to evolve a normalized expression for \( \sigma_{\theta} \).

Evaluating at \( r = b \), equations (32) and (42) can both be written into the form

\[
\sigma_{\theta}(b) = \frac{pD}{4\pi E} b^k \]

where

\[
k_5^{(C.L.M.P.)} = \mu + (1+\mu)\frac{k\gamma}{\gamma^2} + \frac{\gamma^2}{1-\gamma^2} \left[ (1+\gamma) - \frac{1-\mu}{\gamma^2} \right]
\]

and

\[
k_5^{(S.S.)} = (1-\mu) \left[ \frac{1-\gamma}{(1+\mu) + (1-\mu)\gamma^2} - \frac{1}{2} \right]
\]

\[
+ \left( \frac{1+\mu}{2} \right) \left[ \frac{2(1+\mu)k\gamma - (1-\mu)(1-\gamma^2)}{(1+\mu) + (1-\mu)\gamma^2} \right]
\]

Equations (63) and (64) present normalized values of the facing circumferential bending stresses at the insert rim for clamped and simply supported outer edges, respectively.

**NUMERICAL CALCULATIONS AND DESIGN CURVES**

Equations (47) through (63) have presented suitably defined design coefficients \( k_1 \) through \( k_5 \), giving normalized critical values of plate bending and shear deflections, core shear stress, and facing radial and circumferential bending stresses. The quantities associated with bending have been presented in terms of the radius ratio \( b/a \), while those associated with shear have been presented in terms of \( b/a \) and the parameter \( \alpha a \).

Plots of the design coefficient \( k_1 \) for the clamped and simply supported edge conditions, as given by equations (48) and (49), respectively, are provided in figure 2. Poisson's ratio \( \mu \), associated with the facing material, was taken to be 1/3 in this and in all subsequent calculations.
Figure 2.--Normalized maximum plate bending deflection (clamped and simply supported edge).
Figures 3 and 4 present families of curves for the design coefficient $k_2$ for clamped and simply supported outer edges, as provided by equations (52) and (53).

Figures 5 and 6 picture the distribution of normalized core shear stress from the insert rim to the plate outer edge for each outer edge condition when \( aa = 40 \), corresponding to \( \frac{b}{a} = 0.50 \) and 0.10, respectively. These data were obtained from equations (57) and (58).

Figure 7 gives a family of curves for the normalized maximum core shear stress $k_3$ within the clamped plate. These calculations were performed by first determining via a numerical iteration scheme the value of $\beta$ satisfying equation (54) on the interval $gA < \beta < A$ and then utilizing this root in equation (57). Figure 8 presents similar data for the simply supported plate, obtained from equations (55) and (58) in exactly the same manner.

Figures 9 and 10 depict normalized radial and circumferential stress distributions within the facings from the insert rim to the plate outer edge, for both outer edge conditions. Curves for the radial stress distribution were obtained by modifying equations (30) and (40) into the form

$$
\sigma_{R}(r) = -\frac{PD}{4\pi D_t h^2} R(r)
$$

where

$$
R(\text{CLMP}) = -1 + (1+\mu)\frac{a}{r} + \left[\frac{1 - \mu}{(r/a)^2}\right] \frac{a^2}{\gamma - 1}
$$

and

$$
R(\text{S.S.}) = \frac{\mu - 1}{2} + (1+\mu)\frac{b}{r}
\left[\frac{1 - (1+\mu)\gamma^2}{1+\mu} + (1-\mu)\gamma^2\right] - \frac{1 + \mu}{2}\frac{2(1+\mu)\gamma^2}{(1+\mu) + (1-\mu)\gamma^2}
$$

Curves for the circumferential stress distribution were obtained by modifying equations (32) and (42) into the form

$$
\sigma_{\theta}(r) = -\frac{PD}{4\pi D_r h^2} \Theta(r)
$$

where

$$
(\Theta)^{\text{(CLMP)}} = -\mu + (1+\mu)\frac{a}{r} + \frac{2}{\gamma - 1} \left[\frac{1 - \mu}{(r/a)^2}\right] \frac{a^2}{\gamma}
$$

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and

\[
\Phi(S,S_s) = \frac{1 - \mu}{2} \cdot \frac{1 + \mu}{\pi} \left[ \frac{1 - \mu}{(1 + \mu)^2} \right] \cdot \frac{1 - \mu}{(1 + \mu)^2 + (1 - \mu)^2} \cdot \left( \frac{2(1 + \mu)^2 \gamma - (1 - \mu)(1 - \gamma^2)}{(1 + \mu)^2 + (1 - \mu)^2} \right)
\]

(70)

Figure 11 depicts the normalized maximum radial bending stresses, occurring at the insert rim, for each edge condition, as provided by equations (60) and (61). Figure 12 presents the normalized circumferential bending stresses developed at the insert rim for each outer edge condition, utilizing equations (63) and (64).

Inspection of the data presented in the curves of figures 2 through 11, illustrate numerous points worthy of further, more detailed comment. First of all, figure 2 reveals that the maximum bending deflection of a simply supported plate having no insert (i.e., \(\gamma \rightarrow 0\)) is 2.5 times that of a clamped plate. This should be in agreement with corresponding results for homogeneous plates, since \(w_b\) represents the total sandwich deflection when the core is rigid (i.e., when \(G\) and, hence, \(\alpha\) approach infinity). Comparison with Timoshenko reveals this agreement. According to Timoshenko, the center deflections of clamped and simply supported homogeneous plates are related by

\[
\frac{\psi_{\text{max}}}{\psi_{\text{max}}} = \frac{3 + \alpha \omega}{1 + \alpha \omega} \quad \text{(CLNP)}
\]

which, for \(\alpha = \frac{1}{\gamma}\), becomes

\[
\frac{\psi_{\text{max}}}{\psi_{\text{max}}} = 2,5 \omega_{\text{max}} \quad \text{(CLNP)}
\]

in exact agreement with the corresponding sandwich plate results.

Furthermore, comparison of the data of figure 2 with corresponding data for homogeneous circular plates with inserts, provided by Timoshenko reveals agreement between results for the maximum deflections at the insert rim corresponding to both edge conditions.

\[\ \]

\[1\] Hospital's rule was utilized to obtain these results as a limiting process.

Figure 3.--Normalized maximum plate shear deflection (clamped edge).
Figure 4.--Normalized maximum plate shear deflection (simply supported edge).
Figure 5.--Distribution of normalized core shear stress ($\alpha = 40$, $b/a = 0.50$).
Figure 6.--Distribution of normalized core shear stress (\( \alpha \frac{a}{b} = 40, \frac{b}{a} = 0.10 \)).
Figure 7.—Normalized maximum core shear stress (clamped edge).
Figure 8.—Normalized maximum core shear stress (simply supported edge).
Figure 9.--Normalized radial and circumferential facing stress distributions (clamped and simply supported edge; b/a = 0.50).
Figure 10.--Normalized radial and circumferential facing stress distributions (clamped and simply supported edge; b/a = 0.05).
Figure 11. -- Normalized maximum radial facing stress (clamped and simply supported edge).
Figure 12.--Normalized circumferential facing stress at insert rim (clamped and simply supported edge).
The shear deflection $w_s$, as pictured by figures 3 and 4, reveals little difference due to plate outer edge condition, except for very small plates (i.e., small values of $\alpha a$).

Inspection of the core shear stress distributions pictured in figures 5 and 6 reveals that the plate outer edge condition influences core shear stress only in the region near to the outer edge. Figures 7 and 8 show that maximum core shear stress is influenced by the plate outer edge condition only in very small plates, for which simple supports result in slightly higher maximum core shear stress.

The facing radial and circumferential stress distributions pictured in figures 9 and 10 reveal that larger bending stresses are developed in simply supported plates than in clamped plates. Maximum radial stress occurs at the insert rim. Maximum circumferential stress occurs at some location near the insert rim, though the maximum value of $\sigma_{\theta b}$ differs little from that occurring at the insert rim. Both bending stresses are compressive at all points throughout the top facing in the simply supported plate, while they attain both compressive and tensile values in the clamped plate. Inspection of the bending stresses at the insert rim, provided by figures 11 and 12, reveal that the magnitudes of the radial and circumferential stresses there are influenced identically by the plate outer edge condition. Figure 13 illustrates this effect by showing how the radial and circumferential bending stresses at the insert rim are increased in the simply supported plate over those of the clamped plate. Finally, the maximum radial stress at the insert rim agrees with data provided by Timoshenko for homogeneous circular plates with clamped inserts.

**SUMMARY**

An analytical solution, following the theory of Ericksen, has been developed for the bending of circular sandwich plates by a normal load applied through a rigid clamped circular insert. Plates having both clamped and simply supported outer edges have been included in the analysis.

Explicit expressions have been derived for the plate bending and shear deflections, the core shear stress, and the facing radial and circumferential bending stresses corresponding to each outer edge condition. Normalized expressions for the critical values of these quantities have also been derived and presented in a series of design curves.

Comparison has been made with existing theories for equivalent bending of homogeneous plates, and the solutions herein have been found to converge to corresponding homogeneous plate solutions in the proper sense.
Figure 13.--Effect of edge condition on radial and circumferential facing stresses at the insert rim.
**NOTATION**

- $r$: radial position measured from the center of the plate
- $b$: insert radius
- $a$: plate outer radius
- $P$: point load
- $w$: total plate deflection
- $w_b$: plate bending deflection
- $w_s$: plate shear deflection
- $t_1$: thickness of top facing
- $t_2$: thickness of bottom facing
- $h$: distance separating facing midplanes
- $E_{1,2}$: Young's moduli for facing materials
- $\mu$: Poisson's ratio for facing materials
- $\lambda$: 
  \[
  \frac{E_1 t_1^3 + E_2 t_2^3}{12 \lambda}
  \]
- $D_F$: 
  \[
  \frac{E_1 t_1 E_2 t_2 h^2}{(E_1 t_1 + E_2 t_2) \lambda}
  \]
- $D_m$: 
  \[
  D_F + D_m
  \]
- $G$: core modulus of rigidity
- $\alpha$: modified Bessel functions of the first and second kinds, respectively, and of order $\nu$
- $I_{\nu}, K_{\nu}$: core shear stress
- $\sigma_r$: facing radial bending stress
- $\sigma_{r_b}$: contribution to $\sigma_r$ due to bending
- $\sigma_{r_s}$: contribution to $\sigma_r$ due to shear
- $\sigma_\theta$: facing circumferential bending stress
- $\sigma_{\theta_b}$: contribution to $\sigma_\theta$ due to bending
- $\sigma_{\theta_s}$: contribution to $\sigma_\theta$ due to shear
- $\gamma$: $b/a$
- $A$: $\alpha a$
- $k_1$: normalized maximum bending deflection
- $k_2$: normalized maximum shear deflection
- $k_3$: normalized maximum core shear stress
- $k_4$: normalized maximum facing radial bending stress, at insert rim
- $k_5$: normalized facing circumferential bending stress, at insert rim.