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Asymptotically Efficient Estimation of a Bivariate Gaussian–Weibull Distribution and an Introduction to the Associated Pseudo-truncated Weibull

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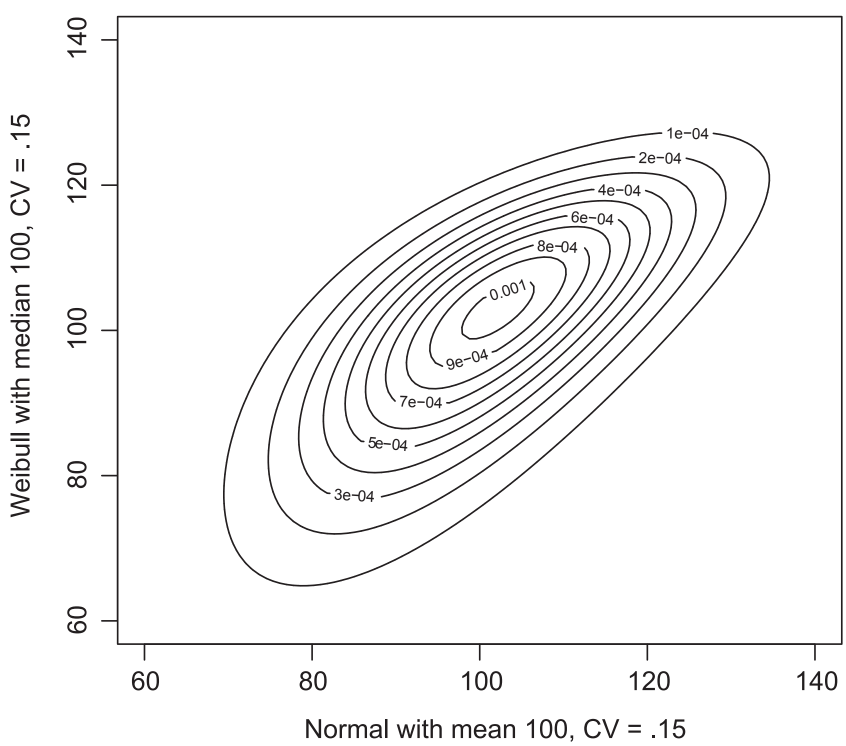

Abstract

Two important wood properties are stiffness (modulus of elasticity or MOE) and bending strength (modulus of rupture or MOR). In the past, MOE has often been modeled as a Gaussian and MOR as a lognormal or a two- or three-parameter Weibull. It is well-known that MOE and MOR are positively correlated. To model the simultaneous behavior of MOE and MOR for the purposes of wood system reliability calculations, we introduce a bivariate Gaussian–Weibull distribution and the associated pseudo-truncated Weibull. We use asymptotically efficient likelihood methods to obtain an estimator of the parameter vector of the bivariate Gaussian--Weibull, and then obtain the asymptotic distribution of this estimator.

Keywords: Reliability, modulus of rupture, modulus of elasticity, one-step Newton estimator, Gaussian copula, normal distribution, Weibull distribution, likelihood methods

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Contents

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1 Introduction

Two important wood properties are stiffness (modulus of elasticity or MOE) and bending strength (modulus of rupture or MOR). In the past, MOE has often been modeled as a Gaussian and MOR as a lognormal or a two- or three-parameter Weibull. (See, for example, ASTM 2010a, Evans and Green 1988, and Green and Evans 1988.)

Design engineers must ensure that the loads to which wood systems are subjected rarely exceed the systems' strengths. To this end ASTM D 2915 (ASTM 2010a), and ASTM D 143 or ASTM D 1990 (ASTM 2010b,c) describe the manner in which "allowable properties" are assigned to populations of structural lumber. In essence, an allowable strength property is calculated by estimating a fifth percentile of a population (actually a 95% content, one-sided lower 75% tolerance bound) and then dividing that value by "duration of load" (aging) and safety factors. The intent is that the population can only be used in applications in which the load does not exceed the allowable property. Of course there are stochastic issues associated with variable loads, uncertainty in estimation, and the division of a percentile with no consideration of population variability. Thus, from a statistician's perspective, this is not an ideal approach to ensuring reliability of wood systems. However, it is the currently codified approach.

To apply this approach, one must obtain estimates of the fifth percentiles of MOR distributions. Currently, one method for obtaining estimates involves fitting a two-parameter Weibull distribution to a sample of MORs. To obtain this fit, either a maximum likelihood approach or a linear regression approach based on order statistics is permitted under ASTM D 5457 (ASTM 2010d).

Unfortunately, these methods are often applied to populations that are not really distributed as two-parameter Weibulls. For example, in the United States, construction grade 2 by 4's are often classified into visual categories—select structural, number 1, number 2—or into machine stress-rated (MSR) grades. In the case of MSR grades, MOE boundaries are selected, MOE is measured nondestructively, and boards are placed into categories based upon the MOE bins into which the boards fall. Because MOE and MOR are correlated, bins with higher MOE boundaries also tend to contain board populations with higher MOR values. The fifth percentiles of these MOR populations are sometimes estimated by fitting Weibull distributions to these populations. Statisticians recognize that this poses a problem. Even if the full population of lumber strengths were distributed as a Weibull, we would not expect that subpopulations formed by visual grades or MOE binning would continue to be distributed as Weibulls.

In fact, such a subpopulation is not distributed as a Weibull. Instead, if the full joint MOE–MOR population were distributed as a bivariate Gaussian–Weibull, the subpopulation would be distributed as a "pseudo-truncated Weibull" (PTW). In this paper, we obtain the distribution of a PTW and show how to obtain estimates of its parameters and its quantiles by fitting a bivariate Gaussian–Weibull to the full MOE–MOR distribution. To do this, we first define a particular form of a bivariate Gaussian–Weibull distribution. In Sections 2 and 3 of this paper, we describe this form and establish that it can be fit by asymptotically efficient likelihood methods in the full MOE–MOR case. In Sections 4 and 5, we discuss the truncated case and derive the density of a PTW.

In a subsequent paper, we will describe a Web-based computer program that we have developed to perform the asymptotically efficient fit. We will use a related program to fit simulated MOE–MOR data and evaluate the small sample quality of the fits. In a third paper, we will show that Weibull fits to PTW data can yield poor estimates of probabilities of failure.

As an aside, we remark that the bivariate Gaussian–Weibull distribution has uses other than as a generator of pseudo-truncated Weibulls. For example, engineers who are interested in simulating the performance of wood systems must begin with a model for the joint stiffness, strength distribution of the members of the system. Provided that we are considering the *full* population, a Gaussian–Weibull is one possible model for this joint distribution.

Bivariate Gaussian–Weibull distributions have not yet appeared in the literature. However, Gumbel (1960), Freund (1961), Marshall and Olkin (1967), Block and Basu (1974), Clayton (1978), Lee (1979), Hougaard (1986), Sarker (1987), Lu and Bhattacharyya (1990), Patra and Dey (1999), Johnson *et al.* (1999), and others have previously investigated bivariate Weibulls.

We note that the bivariate Gaussian–Weibull distribution that we investigate in the current paper is not the only possible bivariate distribution with Gaussian and Weibull marginals. In essence we begin with a "Gaussian copula"—a bivariate uniform distribution generated by starting with a bivariate normal distribution and then applying normal cumulative distribution functions to its marginals. However, there is a large literature on alternative copulas (multivariate distributions with uniform marginals). See, for example, Nelsen (1999) and Jaworski (2010). (Also see Wang *et al.* (2008) for an application of copulas to joint models of tree heights and diameters.) These alternatives would lead to alternative bivariate Gaussian–Weibulls. Ultimately, the test of the usefulness of our proposed version of a Gaussian–Weibull for a particular application will depend on the match between the theoretical distribution and data. Still, we believe that the analysis of our proposed version in the current paper represents a useful step in the construction and evaluation of bivariate Gaussian–Weibull distributions.

2 A bivariate Gaussian–Weibull distribution

To generate a bivariate Gaussian–Weibull distribution, we follow Johnson and Kotz (1972). (Taylor and Bender (1988, 1989) introduced this technique in a lumber context.) That is, let X_1, X_2 be distributed as independent N(0,1)'s. Define $X = \mu + \sigma X_1$ and $Y = \rho X_1 + \sqrt{1-\rho^2}X_2$. Then X is distributed as a $\mathrm{N}(\mu, \sigma^2)$, Y is distributed as a N(0,1), and their correlation is ρ. Now let $U = \Phi(Y)$. Then U is a Uniform(0,1) random variable that is correlated with X. Finally, let $W = (-\log(1-U))^{1/\beta}/\gamma$. Then W is distributed as a Weibull with shape parameter β and scale parameter $1/\gamma$, and the pair X, W have our joint "bivariate Gaussian–Weibull" distribution. *In this paper, we require that* $\beta > 1$. Given this generating process, it is straightforward to show (see Appendix A) that the joint density is given by

$$f(x, w) = \mathrm{bivnorm}(x, y) \times \mathrm{weib}(w)/\phi(y)$$

where ϕ is the N(0,1) probability density function, Φ is the N(0,1) cumulative distribution function,

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma \times w)^\beta\right)\right)$$

$$\mathrm{weib}(w) = \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma \times w)^\beta\right)$$

and

$$\mathrm{bivnorm}(x, y) = \frac{1}{2\pi}\frac{1}{\sigma\sqrt{1-\rho^2}}\exp(-\mathrm{arg})$$

where

$$\mathrm{arg} = \left((x-\mu)^2/\sigma^2 - 2\rho(x-\mu)y/\sigma + y^2\right)/(2(1-\rho^2))$$

In Figures 1–9 we provide contour plots of the bivariate Gaussian–Weibull distribution for coefficients of variation (CV's) equal to 0.35, 0.25, and 0.15, and generating correlations equal to 0.5, 0.7, and 0.9. The CV of a univariate distribution is its standard deviation divided by its mean. Note that as the CV declines from 0.35 to 0.25 to 0.15 (as the Weibull shape parameter increases from 3.13 to 4.54 to 7.91) the density contours become much less elliptical. That is, the distribution diverges from a bivariate normal. We would expect this as a Weibull is "like a normal" for shape near 3.6 (skewness equals 0.00056, excess kurtosis equals -0.28), and a Weibull becomes skewed to the left and leptokurtic as the shape increases.

3 Asymptotic distribution of the estimated parameter vector of the bivariate Gaussian–Weibull distribution

Now assume that we have have n independent pairs of observations, $(x_1, w_1), \ldots, (x_n, w_n)$ from the bivariate Gaussian–Weibull distribution. Then we have the following theorem.

Theorem 1

$$\sqrt{n}\left(\begin{pmatrix}\hat{\mu}\\ \hat{\sigma}\\ \hat{\rho}\\ \hat{\gamma}\\ \hat{\beta}\end{pmatrix} - \begin{pmatrix}\mu\\ \sigma\\ \rho\\ \gamma\\ \beta\end{pmatrix}\right) \xrightarrow{D} \mathrm{N}(\mathbf{0}, I(\boldsymbol{\theta})^{-1})$$

where $\boldsymbol{\theta} \equiv (\mu, \sigma, \rho, \gamma, \beta)^T$, $\hat{\mu}$, $\hat{\sigma}$ are one-step Newton estimators based on the standard univariate normal maximum likelihood estimators of the mean and standard deviation of a Gaussian, $\hat{\gamma}$, $\hat{\beta}$ are one-step Newton estimators based on the standard maximum likelihood estimators of 1/scale and shape for a Weibull, $\hat{\rho}$ is a one-step Newton estimator based on the $\sqrt{n}$-consistent estimator of ρ introduced in Appendix B, and the elements of $I(\boldsymbol{\theta})$ are listed in Appendix E2 (and E3).

Proof

The proof is an application of theorem 4.2 of chapter 6 of Lehmann (1983). To invoke Lehmann's theorem we must establish a series of conditions.

Lehmann's conditions (A0)–(A2)

That conditions (A0)–(A2) hold is clear.

Lehmann's condition (A)

That condition (A) holds (the existence of third partials of the density) is clear.

Lehmann's condition (B)

Lehmann's condition (B)(8) is established in Appendix E1. Lehmann's condition (B)(9) is established in Appendices E2 and E3.

Lehmann's condition (C)

The fact that the information matrix is positive definite is established in Appendix I.

Lehmann's condition (D)

Lehmann's condition (D) is established in Appendix J. ■

A Web program that obtains the asymptotically efficient likelihood estimates of the five parameters and confidence bounds on these estimates will be described in a subsequent paper.

4 A truncated bivariate Gaussian–Weibull distribution

In wood engineering applications, it is often the case that we do not have data from a full bivariate Gaussian–Weibull distribution. Instead, we have data from the subpopulation that is formed by considering lumber whose MOE values lie between two pre-determined limits, c_{l} and c_{u} (that is, we have machine stress-rated lumber). It is clear that the joint density in this case is

$$f(x,w)/\left(\Phi((c_{\mathrm{u}}-\mu)/\sigma)-\Phi((c_{\mathrm{l}}-\mu)/\sigma)\right) \tag{1}$$

for x between c_{l} and c_{u} and 0 elsewhere.

5 The pseudo-truncated Weibull distribution

The pseudo-truncated Weibull distribution function at w is given by integrating the truncated bivariate Gaussian–Weibull density (1) over the region $[c_{\mathrm{l}}, c_{\mathrm{u}}] \times [0, w]$. That is (from result (11) in Appendix A)

$$F_{\mathrm{PTW}}(w) = \int_0^w F_1(s) \times F_2(s)/\left(\Phi((c_{\mathrm{u}}-\mu)/\sigma)-\Phi((c_{\mathrm{l}}-\mu)/\sigma)\right) ds \tag{2}$$

where

$$F_1(s) \equiv \gamma^\beta \beta s^{\beta-1} \exp\left(-(\gamma s)^\beta\right) \tag{3}$$

and

$$F_2(s) \equiv \int_{c_{\mathrm{l}}}^{c_{\mathrm{u}}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{1-\rho^2}} \exp\left(-\left((x-\mu)/\sigma-\rho y\right)^2/(2(1-\rho^2))\right) dx \tag{4}$$

where

$$y = \Phi^{-1}\left(1-\exp\left(-(\gamma s)^\beta\right)\right)$$

We have

$$\begin{aligned}
F_2 &= \int_{(c_{\mathrm{l}}-\mu)/\sigma}^{(c_{\mathrm{u}}-\mu)/\sigma} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\left(x/\sqrt{1-\rho^2}-\rho y/\sqrt{1-\rho^2}\right)^2/2\right) dx \\
&= \int_{(c_{\mathrm{l}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)}^{(c_{\mathrm{u}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)} \frac{1}{\sqrt{2\pi}} \exp\left(-\left(x-\rho y/\sqrt{1-\rho^2}\right)^2/2\right) dx \qquad (5) \\
&= \int_{(c_{\mathrm{l}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}}^{(c_{\mathrm{u}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}} \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) dx \\
&= \Phi\left((c_{\mathrm{u}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}\right) - \Phi\left((c_{\mathrm{l}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}\right)
\end{aligned}$$

From results (2), (3), and (5), the pseudo-truncated Weibull density is given by

$$\begin{aligned}
f_{\mathrm{PTW}}(w) &= \frac{d}{dw} F_{\mathrm{PTW}}(w) \qquad (6) \\
&= \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \\
&\quad \times \left(\Phi\left((c_{\mathrm{u}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}\right) - \Phi\left((c_{\mathrm{l}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}\right)\right) \\
&\quad \div \left(\Phi((c_{\mathrm{u}}-\mu)/\sigma) - \Phi((c_{\mathrm{l}}-\mu)/\sigma)\right)
\end{aligned}$$

where

$$y = \Phi^{-1}\left(1-\exp\left(-(\gamma w)^\beta\right)\right)$$

Thus, as we would expect, for $\rho = 0$, the pseudo-truncated Weibull density is simply the Weibull density, $\gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)$. In Appendix K, we show that as $\rho \to 1$, the pseudo-truncated Weibull density converges to 0 for w below w_l or w above w_u where w_l is defined by

$$\Phi\left(\frac{c_l-\mu}{\sigma}\right) = 1-\exp\left(-(\gamma w_l)^\beta\right) \qquad (7)$$

and w_u is defined by

$$\Phi\left(\frac{c_u-\mu}{\sigma}\right) = 1-\exp\left(-(\gamma w_u)^\beta\right) \qquad (8)$$

In Appendix K, we also show that for $w \in (w_l, w_u)$, as $\rho \to 1$, the pseudo-truncated Weibull density converges to

$$\gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) / \left(\exp\left(-(\gamma w_l)^\beta\right) - \exp\left(-(\gamma w_u)^\beta\right)\right)$$

That is, as $\rho \to 1$, the density of a pseudo-truncated Weibull converges to the density of a truncated Weibull.

Figures 10 and 11 are Weibull probability plots of PTW data. That is, we plot the ordered data from a PTW sample against the predicted ordered data from the best Weibull fit to the data. If the data really were Weibull, then the plots would be approximately linear. In Figure 10, the generating X, Y correlation was 0, so the data actually was Weibull and the plot is approximately linear. In Figure 11, the generating X, Y correlation was 0.99, so the data was "far from Weibull" and the plot is quite nonlinear. For both data sets, the Weibull coefficient of variation was 0.25 and c_{l} and c_{u} corresponded to the 0.2 and 0.8 quantiles of the Gaussian distribution.

In Appendix L, we formally establish that pseudo-truncated Weibull distributions are not Weibull distributions.

6 Summary

In the context of wood strength modeling, we have introduced a bivariate Gaussian–Weibull distribution and the associated pseudo-truncated Weibull distribution. In this paper, we obtain the asymptotic distribution of the estimated parameter vector for a bivariate Gaussian–Weibull distribution. In a subsequent paper, we will discuss a Web-based computer program that implements this theory to obtain estimates of the parameters of a bivariate Gaussian–Weibull distribution. In a third paper, we will investigate the question of whether allowable property estimates based on a Weibull assumption can be poor if the strength population is actually a pseudo-truncated Weibull population.

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8 Appendix A—Bivariate Gaussian–Weibull density

Let X, Y have a joint bivariate normal distribution with

$$X \sim \mathrm{N}(\mu, \sigma^2)$$

$$Y \sim \mathrm{N}(0, 1)$$

and correlation$(X, Y) = \rho$.

Since $Y \sim \mathrm{N}(0, 1)$, we know that $\Phi(Y)$ is distributed as a Uniform(0,1). (Here, Φ denotes the N(0,1) cumulative distribution function.) Thus, we know that

$$W \equiv (-\log(1 - \Phi(Y)))^{1/\beta} / \gamma \sim \mathrm{Weibull}(\gamma, \beta) \tag{9}$$

(a two-parameter Weibull distribution with scale parameter $1/\gamma$ and shape parameter β).

We then say that X, W have a bivariate Gaussian–Weibull distribution with parameters μ, σ, ρ, γ, and β.

Using the multivariate form of the change-of-variables theorem (see, for example, Rudin 1987), we can calculate the joint density function of X, W. First, we invert Equation (9) to obtain

$$Y = \Phi^{-1}\left(1 - \exp\left(-(\gamma \times W)^\beta\right)\right)$$

Thus, the transform that takes (x, w) to (x, y) is

$$\mathbf{T}(x, w) = \begin{pmatrix} T_1(x, w) \\ T_2(x, w) \end{pmatrix} = \begin{pmatrix} x \\ \Phi^{-1}\left(1 - \exp(-(\gamma \times w)^\beta)\right) \end{pmatrix}$$

The corresponding Jacobian matrix is

$$\begin{pmatrix} \partial T_1/\partial x & \partial T_1/\partial w \\ \partial T_2/\partial x & \partial T_2/\partial w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^\beta \beta w^{\beta-1} \exp(-(\gamma \times w)^\beta)/\phi\left(\Phi^{-1}\left(1 - \exp(-(\gamma \times w)^\beta)\right)\right) \end{pmatrix}$$

and the absolute value of its determinant is

$$\det = \gamma^\beta \beta w^{\beta-1} \exp(-(\gamma \times w)^\beta)/\phi\left(\Phi^{-1}\left(1 - \exp(-(\gamma \times w)^\beta)\right)\right)$$

Thus, the Gaussian–Weibull probability density function (pdf) at x, w is

$$\mathrm{bivnorm}(x, y, \mu, \sigma, \rho) \times \det \tag{10}$$

where

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma \times w)^\beta\right)\right)$$

and

$$\mathrm{bivnorm}(x, y, \mu, \sigma, \rho) = \frac{1}{2\pi} \times \frac{1}{\sigma\sqrt{1 - \rho^2}} \times \exp(\mathrm{arg})$$

where

$$\begin{aligned} \mathrm{arg} &= -\left((x - \mu)^2/\sigma^2 - 2\rho(x - \mu)y/\sigma + y^2\right)/(2(1 - \rho^2)) \\ &= -\left((x - \mu)^2/\sigma^2 - 2\rho(x - \mu)y/\sigma + \rho^2 y^2 + y^2 - \rho^2 y^2\right)/(2(1 - \rho^2)) \\ &= -((x - \mu)/\sigma - \rho y)^2/(2(1 - \rho^2)) - y^2/2 \end{aligned}$$

That is, the Gaussian–Weibull pdf at x, w is given by

$$\begin{aligned} \mathrm{gaussweib}(x, w; \mu, \sigma, \rho, \gamma, \beta) \equiv\ & \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \\ & \times \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma\sqrt{1 - \rho^2}} \exp\left(-((x - \mu)/\sigma - \rho y)^2/(2(1 - \rho^2))\right) \end{aligned} \tag{11}$$

9 Appendix B—$\sqrt{n}$-Consistent Initial Estimators of the Parameters

We will need the following two lemmas in our development. The first of these lemmas provides a useful fact about the tail behavior of normal distributions. Versions of this fact have appeared previously in the statistical literature. See, for example, the discussions of "Mills' ratio" in Kendall and Stuart (1977) and Johnson and Kotz (1970). The particular form of the fact described in Lemma 1 is due to Gordon (1941). A simple proof of Lemma 1 is given in Verrill and Durst (2005).

Lemma 1

For $x < 0$,

$$x^2/(x^2+1) < \Phi(x)/\left(\phi(x)/(-x)\right) < 1 \tag{12}$$

and for $x > 0$,

$$x^2/(x^2+1) < (1-\Phi(x))/(\phi(x)/x) < 1 \tag{13}$$

where $\Phi(x)$ is the N(0,1) cumulative distribution function and $\phi(x)$ is the N(0,1) probability density function.

Lemma 2

Let $W \sim \text{Weibull}(\gamma, \beta)$. Then

$$E\left((\log(W))^2\right) < \infty$$

Proof

We have

$$\begin{aligned}\int_0^\infty (\log(w))^2\,\gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \quad &\le \quad \gamma^\beta\beta\int_0^1 (\log(w))^2 w^{\beta-1}dw \\ &\quad + \int_0^\infty w^2\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw\end{aligned} \tag{14}$$

We know from Weibull theory that the second term on the right hand side in (14) is finite. Consider the first integral on the right hand side.

$$\begin{aligned}\int_0^1 (\log(w))^2 w^{\beta-1}dw &= \int_{-\infty}^0 w^2\exp(w(\beta-1))\exp(w)dw \qquad (15)\\ &= \int_{-\infty}^0 w^2\exp(w\beta)dw \\ &= (1/\beta)\int_0^\infty w^2\beta\exp(-w\beta)dw \\ &= 2/\beta^3 \qquad (16)\end{aligned}$$

■

A referee has noted that

$$E\left(\log(W)\right) = -\left(\gamma_{\mathrm{E}} + \beta\log(\gamma)\right)/\beta \tag{17}$$

and

$$E\left((\log(W))^2\right) = \frac{\pi^2}{6\beta^2} + \left(E\left(\log(W)\right)\right)^2 \tag{18}$$

where γ_{E} is the Euler-Mascheroni constant. Result (17) follows from

$$\begin{aligned}\int_0^\infty \log(w)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)dw &= \int_0^\infty \log\left(s^{1/\beta}\right)\gamma^\beta\beta s^{(\beta-1)/\beta}\exp\left(-(\gamma^\beta s)\right)\frac{1}{\beta}s^{1/\beta-1}ds \\ &= \left(\gamma^\beta/\beta\right)\int_0^\infty \log(s)\exp\left(-(\gamma^\beta s)\right)ds \\ &= \frac{1}{\beta}\left(\int_0^\infty \log(t)\exp(-t)dt - \beta\log(\gamma)\right) \\ &= \left(-\gamma_{\mathrm{E}} - \beta\log(\gamma)\right)/\beta \qquad (19)\end{aligned}$$

To invoke theorem 4.2 of Lehmann (1983) to establish that our final estimators of the parameters are asymptotically efficient, we first need to establish that our initial estimates of the parameters are $\sqrt{n}$-consistent. ($\hat{a}_n$ is a $\sqrt{n}$-consistent estimator of a if $\sqrt{n}(\hat{a}_n - a) = O_p(1)$. A sequence of random variables $\{X_n\}$ is $O_p(1)$ if given any $\epsilon > 0$, we can find constants M_ϵ, N_ϵ such that $n > N_\epsilon$ implies that $\mathrm{Prob}(X_n > M_\epsilon) < \epsilon$.) As our initial estimators of μ and σ, we take the standard one-variable estimators $\bar{x} = \sum x_i/n$ and $s = \sqrt{\sum(x_i - \bar{x})^2/(n-1)}$. As our initial estimators of γ and β, we take the one-variable maximum likelihood estimators, $\hat{\gamma}$ and $\hat{\beta}$. Thus, our initial estimators of μ, σ, γ, and β are $\sqrt{n}$-consistent. Our initial estimator of ρ is given by

$$\hat{\rho} \equiv \hat{s}_{xy}/\sqrt{s_{xx} \times \hat{s}_{yy}} \qquad (20)$$

where

$$\begin{aligned}\hat{s}_{xy} &\equiv \sum_{i=1}^n (x_i - \bar{x})(\hat{y}_i - \bar{\hat{y}}) \\ s_{xx} &\equiv \sum_{i=1}^n (x_i - \bar{x})^2 \\ \hat{s}_{yy} &\equiv \sum_{i=1}^n (\hat{y}_i - \bar{\hat{y}})^2 \\ \bar{\hat{y}} &\equiv \sum_{i=1}^n \hat{y}_i/n\end{aligned}$$

and

$$\hat{y}_i \equiv g(w_i; \hat{\gamma}, \hat{\beta}) \equiv \Phi^{-1}\left(1 - \exp\left(-(\hat{\gamma} \times w_i)^{\hat{\beta}}\right)\right) \qquad (21)$$

Theorem 2

$$\sqrt{n}\,(\hat{\rho} - \rho) = O_p(1)$$

where $\hat{\rho}$ is defined in Equation (20).

Proof

Define

$$\begin{aligned}s_{xy} &\equiv \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y}) \\ s_{yy} &\equiv \sum_{i=1}^n (y_i - \bar{y})^2 \\ \bar{y} &\equiv \sum_{i=1}^n y_i/n\end{aligned}$$

where

$$y_i \equiv g(w_i;\gamma,\beta) \equiv \Phi^{-1}\left(1-\exp\left(-(\gamma\times w_i)^{\beta}\right)\right) \tag{22}$$

(The distinction between the "hatted" variables in definitions (21) and the "unhatted" variables in definitions (22) is that in the hatted case, γ, β are replaced by their estimates $\hat{\gamma}, \hat{\beta}$.)

We know that

$$r \equiv s_{xy}/\sqrt{s_{xx}\times s_{yy}}$$

is a $\sqrt{n}$-consistent estimator of ρ. (That is, we know that $\sqrt{n}(r-\rho) = O_p(1)$.) Thus, we will be done if we can show that

$$\sqrt{n}\ (r-\hat{\rho}) = O_p(1) \tag{23}$$

We have

$$\begin{aligned}
r-\hat{\rho} &= s_{xy}/\sqrt{s_{xx}\times s_{yy}} - \hat{s}_{xy}/\sqrt{s_{xx}\times \hat{s}_{yy}} \\
&= \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\sqrt{s_{xx}\times s_{yy}}} - \frac{\sum_{i=1}^{n}(x_i-\bar{x})(\hat{y}_i-\bar{\hat{y}})}{\sqrt{s_{xx}\times s_{yy}}} \\
&\quad + \frac{\sum_{i=1}^{n}(x_i-\bar{x})(\hat{y}_i-\bar{\hat{y}})}{\sqrt{s_{xx}\times s_{yy}}} - \frac{\sum_{i=1}^{n}(x_i-\bar{x})(\hat{y}_i-\bar{\hat{y}})}{\sqrt{s_{xx}\times \hat{s}_{yy}}} \\
&\equiv D_1 + D_2
\end{aligned} \tag{24}$$

To show that $\sqrt{n}D_1 = O_p(1)$, we need to show that

$$\sqrt{n}\left(\sum_{i=1}^{n}(x_i-\bar{x})\left(y_i-\bar{y}-(\hat{y}_i-\bar{\hat{y}})\right)/n\right) = O_p(1) \tag{25}$$

By the Cauchy–Schwarz inequality and the fact that $\sum_{i=1}^{n}(x_i-\bar{x})^2/n \xrightarrow{p} \sigma^2$, we know that we can establish result (25) by establishing that

$$\sum_{i=1}^{n}\left(y_i-\bar{y}-(\hat{y}_i-\bar{\hat{y}})\right)^2 = O_p(1) \tag{26}$$

and it is clear that result (26) follows if

$$\sum_{i=1}^{n}(y_i-\hat{y}_i)^2 = O_p(1) \tag{27}$$

(This follows because $\sum_{i=1}^{n}(z_i-\bar{z})^2 \le \sum_{i=1}^{n} z_i^2$.)

From definitions (21) and (22) we have

$$\sum_{i=1}^{n}(y_i-\hat{y}_i)^2 = \sum_{i=1}^{n}\left(g(w_i;\gamma,\beta)-g(w_i;\hat{\gamma},\hat{\beta})\right)^2 \tag{28}$$

By Taylor's theorem this equals

$$\sum_{i=1}^{n}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}(\hat{\gamma}-\gamma) + \frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\beta}|_{\boldsymbol{\theta}_{*,i}}(\hat{\beta}-\beta)\right)^2 \tag{29}$$

where $\boldsymbol{\theta} = (\gamma,\beta)^T$ and $\boldsymbol{\theta}_{*,i} \equiv (\gamma_{*,i},\beta_{*,i})^T$ lies on the line between $(\gamma,\beta)^T$ and $(\hat{\gamma},\hat{\beta})^T$.

Thus, given the Cauchy–Schwarz inequality, to establish result (27), it is sufficient to establish

$$\sum_{i=1}^{n}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}\right)^2(\hat{\gamma}-\gamma)^2 = O_p(1) \tag{30}$$

and

$$\sum_{i=1}^{n}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\beta}|_{\boldsymbol{\theta}_{*,i}}\right)^2(\hat{\beta}-\beta)^2 = O_p(1) \tag{31}$$

Because $\hat{\gamma}$ and $\hat{\beta}$ are the maximum likelihood estimates of γ and β, to establish results (30) and (31), it is sufficient to establish

$$\sum_{i=1}^{n}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}\right)^2/n = O_p(1) \tag{32}$$

and

$$\sum_{i=1}^{n}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\beta}|_{\boldsymbol{\theta}_{*,i}}\right)^2/n = O_p(1) \tag{33}$$

Consider result (32). We have

$$\begin{aligned}\sum_{i=1}^{n}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}\right)^2/n &= \sum_{w_i<w_{\text{low}}}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}\right)^2/n \\ &\quad+\sum_{w_{\text{low}}\le w_i\le w_{\text{up}}}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}\right)^2/n \\ &\quad+\sum_{w_{\text{up}}<w_i}\left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}\right)^2/n \\ &\equiv S_1+S_2+S_3\end{aligned} \tag{34}$$

where $0 < w_{\text{low}} < w_{\text{up}}$. We will show that S_1, S_2, and S_3 are $O_p(1)$.

First, consider S_2. Let $\epsilon > 0$ be given. Recall that in this paper we require that $\beta > 1$. Because $\hat{\gamma} \xrightarrow{p} \gamma$, $\hat{\beta} \xrightarrow{p} \beta > 1$, $\sum_{i=1}^{n} w_i^r/n \xrightarrow{p} E(W^r)$, and (from Lemma 2) $\sum_{i=1}^{n}(\log(w_i))^2/n \xrightarrow{p} E((\log(W))^2)$, given any $\gamma_{\text{low}}, \gamma_{\text{up}}$ such that $0 < \gamma_{\text{low}} < \gamma < \gamma_{\text{up}}$, $\gamma_{\text{low}} < 1 < \gamma_{\text{up}}$ and any $\beta_{\text{low}}, \beta_{\text{up}}$ such that $1 < \beta_{\text{low}} < \beta < \beta_{\text{up}}$, we can find an N such that $n > N$ implies

$$\text{prob}\,(A_n \cap B_n \cap C_n) > 1-\epsilon \tag{35}$$

where A_n is the set on which

$$\beta_{\text{low}} < \hat{\beta} < \beta_{\text{up}} \text{ and } \gamma_{\text{low}} < \hat{\gamma} < \gamma_{\text{up}}$$

B_n is the set on which

$$\left|\sum_{i=1}^{n} w_i^{2\beta_{\text{up}}}/n - E\left(W^{2\beta_{\text{up}}}\right)\right| < 1$$

and C_n is the set on which

$$\left|\sum_{i=1}^{n}(\log(w_i))^2/n - E\left((\log(W))^2\right)\right| < 1$$

Define

$$
\begin{aligned}
w_{1/4} &\equiv (\log(4/3))^{1/\beta}/\gamma && (36)\\
w_{3/4} &\equiv (\log(4))^{1/\beta}/\gamma && (37)
\end{aligned}
$$

so

$$
\begin{aligned}
1-\exp\left(-(\gamma \times w_{1/4})^{\beta}\right) &= 1/4 && (38)\\
1-\exp\left(-(\gamma \times w_{3/4})^{\beta}\right) &= 3/4 && (39)
\end{aligned}
$$

Now choose w_{low} small enough so that

$$\gamma_{\text{low}} \times w_{\text{low}} < \gamma_{\text{up}} \times w_{\text{low}} < 1,\ (\gamma_{\text{up}} \times w_{\text{low}})^{\beta_{\text{low}}} \quad < \quad \min\left(1/4, (\gamma \times w_{1/4})^{\beta}\right) \tag{40}$$

and choose w_{up} large enough so that

$$w_{\text{up}} > 1,\ \gamma_{\text{up}} \times w_{\text{up}} > \gamma_{\text{low}} \times w_{\text{up}} > 1,\ (\gamma_{\text{low}} \times w_{\text{up}})^{\beta_{\text{low}}} > (\gamma \times w_{3/4})^{\beta} \tag{41}$$

Let

$$K \equiv \min\left(\phi\left(\Phi^{-1}\left(1-\exp(-(\gamma_{\text{low}} \times w_{\text{low}})^{\beta_{\text{up}}})\right)\right), \phi\left(\Phi^{-1}\left(1-\exp(-(\gamma_{\text{up}} \times w_{\text{up}})^{\beta_{\text{up}}})\right)\right)\right)$$

Then, because $\gamma_{\text{low}} \times w_{\text{low}} < 1$ and $\gamma_{\text{up}} \times w_{\text{up}} > 1$, on set A_n we have

$$\phi\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i} \times w_i)^{\beta_{*,i}})\right)\right) \geq K \text{ for all } w_i \text{ in } [w_{\text{low}}, w_{\text{up}}] \tag{42}$$

so

$$
\begin{aligned}
\left|\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial \gamma}|_{\boldsymbol{\theta}_{*,i}}\right| &= \beta_{*,i}\gamma_{*,i}^{\beta_{*,i}-1} w_i^{\beta_{*,i}} \exp\left(-(\gamma_{*,i} w_i)^{\beta_{*,i}}\right)/\phi\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i} w_i)^{\beta_{*,i}})\right)\right)\\
&\leq \beta_{*,i} \times \gamma_{*,i}^{\beta_{*,i}-1} \times w_i^{\beta_{*,i}}/K && (43)\\
&\leq \beta_{\text{up}} \times \gamma_{\text{up}}^{\beta_{\text{up}}-1} \times w_{\text{up}}^{\beta_{\text{up}}}/K \equiv M_1
\end{aligned}
$$

From results (35) and (43), we know that $n > N$ implies that

$$\text{prob}(S_2 < M_1^2) > 1-\epsilon$$

That is,

$$S_2 = O_p(1) \tag{44}$$

Now consider S_1. Let $\epsilon > 0$ be given. As in the S_2 analysis, we can find an N such that $n > N$ implies that result (35) holds. Then, by our previous choice of w_{low} (see (40)), on A_n we have

$$
\begin{aligned}
(\gamma_{*,i} \times w_{\text{low}})^{\beta_{*,i}} &< (\gamma_{\text{up}} \times w_{\text{low}})^{\beta_{*,i}}\\
&< (\gamma_{\text{up}} \times w_{\text{low}})^{\beta_{\text{low}}} && (45)\\
&< \min\left(1/4, (\gamma \times w_{1/4})^{\beta}\right)
\end{aligned}
$$

and thus, for $w_i < w_{\text{low}}$,

$$0 < (\gamma_{*,i} \times w_i)^{\beta_{*,i}} < (\gamma_{*,i} \times w_{\text{low}})^{\beta_{*,i}} < \min\left(1/4, (\gamma \times w_{1/4})^{\beta}\right) \tag{46}$$

so

$$\Phi^{-1}\left(1-\exp\left(-(\gamma_{*,i}\times w_i)^{\beta_{*,i}}\right)\right) < \Phi^{-1}\left(1-\exp\left(-(\gamma\times w_{1/4})^{\beta}\right)\right) = \Phi^{-1}(1/4) < 0 \tag{47}$$

Thus, on A_n, for $w_i < w_{\text{low}}$, we can apply Lemma 1 to obtain

$$\begin{aligned}
\left|\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial \gamma}|_{\boldsymbol{\theta}_{*,i}}\right| &= \beta_{*,i}\gamma_{*,i}^{\beta_{*,i}-1}w_i^{\beta_{*,i}}\exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right)/\phi\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\right) \\
&\le \beta_{*,i}\times\gamma_{*,i}^{\beta_{*,i}-1}\times w_i^{\beta_{*,i}}/\text{den} \qquad (48)\\
&\le (\beta_{\text{up}}/\gamma_{\text{low}})\times(\gamma_{*,i}w_i)^{\beta_{*,i}}/\text{den}
\end{aligned}$$

where

$$\text{den} = -\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\times\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\right) \tag{49}$$

(We now want to show that den goes to 0 no faster than the $(\gamma_{*,i}w_i)^{\beta_{*,i}}$ in the numerator.)

By results (47)–(49), on A_n, for $w_i < w_{\text{low}}$,

$$\left|\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial \gamma}|_{\boldsymbol{\theta}_{*,i}}\right| < -\left((\beta_{\text{up}}/\gamma_{\text{low}})/\Phi^{-1}(1/4)\right)\times(\gamma_{*,i}w_i)^{\beta_{*,i}}/\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right) \tag{50}$$

Now by result (46), on A_n, for $w_i < w_{\text{low}}$, and $x \equiv (\gamma_{*,i}w_i)^{\beta_{*,i}}$,

$$\begin{aligned}
1-\exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right) &= 1-\left(1-x+x^2/2!-x^3/3!+x^4/4!-\ldots\right)\\
&= x\left(1-x/2!+x^2/3!-x^3/4!+\ldots\right)\\
&> x(1-x/2)\\
&> x(1-1/8) = x\times 7/8 \qquad (51)
\end{aligned}$$

or

$$1/\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right) < (8/7)\times(1/(\gamma_{*,i}w_i)^{\beta_{*,i}}) \tag{52}$$

Thus, by results (50) and (52), on A_n, for $w_i < w_{\text{low}}$

$$\left|\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial \gamma}|_{\boldsymbol{\theta}_{*,i}}\right| < -\left((\beta_{\text{up}}/\gamma_{\text{low}})/\Phi^{-1}(1/4)\right)\times(8/7)\equiv M_2 \tag{53}$$

From results (35) and (53), we know that $n > N$ implies that

$$\text{prob}(S_1 < M_2^2) > 1-\epsilon$$

That is,

$$S_1 = O_p(1) \tag{54}$$

Now consider S_3. Let $\epsilon > 0$ be given. As in the S_2 analysis, we can find an N such that $n > N$ implies that result (35) holds. Then, by our previous choice of w_{up} (see (41)), on A_n, for $w_i > w_{\text{up}}$, we have

$$\begin{aligned}
(\gamma_{*,i}\times w_i)^{\beta_{*,i}} &\ge (\gamma_{\text{low}}\times w_{\text{up}})^{\beta_{\text{low}}} \qquad (55)\\
&\ge (\gamma\times w_{3/4})^{\beta}
\end{aligned}$$

so

$$\Phi^{-1}\left(1-\exp\left(-(\gamma_{*,i}\times w_i)^{\beta_{*,i}}\right)\right) > \Phi^{-1}(3/4) > 0 \tag{56}$$

Thus, on A_n, for $w_i > w_{\rm up}$, we can apply Lemma 1 to obtain

$$\begin{aligned}
\left|\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}\right| &= \beta_{*,i}\gamma_{*,i}^{\beta_{*,i}-1}w_i^{\beta_{*,i}}\exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right)/\phi\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\right) \\
&\leq \beta_{*,i}\times\gamma_{*,i}^{\beta_{*,i}-1}\times w_i^{\beta_{*,i}}\times\exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right)/\text{den} \qquad (57)\\
&\leq \beta_{\rm up}\times\gamma_{\rm up}^{\beta_{\rm up}-1}\times w_i^{\beta_{\rm up}}\times\exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right)/\text{den}
\end{aligned}$$

where

$$\text{den} = \exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right)\times\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\right) \qquad (58)$$

By results (56)–(58), on A_n, for $w_i > w_{\rm up}$,

$$\left|\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\gamma}|_{\boldsymbol{\theta}_{*,i}}\right| \leq \beta_{\rm up}\times\gamma_{\rm up}^{\beta_{\rm up}-1}\times w_i^{\beta_{\rm up}}/\Phi^{-1}(3/4) \qquad (59)$$

Thus, on $A_n\cap B_n$,

$$\begin{aligned}
S_3 &\leq \left(\beta_{\rm up}\times\gamma_{\rm up}^{\beta_{\rm up}-1}/\Phi^{-1}(3/4)\right)^2\times\sum_{i=1}^n w_i^{2\beta_{\rm up}}/n \qquad (60)\\
&< \left(\beta_{\rm up}\times\gamma_{\rm up}^{\beta_{\rm up}-1}/\Phi^{-1}(3/4)\right)^2\times\left(E\left(W^{2\beta_{\rm up}}\right)+1\right)\equiv M_3 \qquad (61)
\end{aligned}$$

From results (35) and (60), we know that $n > N$ implies that

$$\text{prob}(S_3 < M_3) > 1-\epsilon$$

That is,

$$S_3 = O_p(1) \qquad (62)$$

Results (34), (44), (54), and (62) establish result (32). Thus to complete the proof of (27), we need to establish result (33).

In general, the proof of result (33) is essentially the same as the proof of result (32). However, there is one distinction that must be addressed. We have

$$\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\beta}|_{\boldsymbol{\theta}_{*,i}} = (\gamma_{*,i}w_i)^{\beta_{*,i}}\times\log(\gamma_{*,i}w_i)\times\exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right)/\phi\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\right) \qquad (63)$$

Given this equality, it is clear that the analogues to the results $S_2 = O_p(1)$ and $S_3 = O_p(1)$ follow as they did in the proof of result (32). However, the analogue of $S_1 = O_p(1)$ requires slightly more care. In particular, on A_n, for $w_i < w_{\rm low}$, we can apply Lemma 1 to obtain

$$\begin{aligned}
\left|\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial\beta}|_{\boldsymbol{\theta}_{*,i}}\right| &= (\gamma_{*,i}w_i)^{\beta_{*,i}}\times|\log(\gamma_{*,i}w_i)|\times\exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right)/\phi\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\right)\\
&\leq (\gamma_{*,i}w_i)^{\beta_{*,i}}\times|\log(\gamma_{*,i}w_i)|\times\exp\left(-(\gamma_{*,i}w_i)^{\beta_{*,i}}\right)/\text{den} \qquad (64)
\end{aligned}$$

where

$$\text{den} = -\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\times\left(\Phi^{-1}\left(1-\exp(-(\gamma_{*,i}w_i)^{\beta_{*,i}})\right)\right) \qquad (65)$$

By results (47), (52), (64) and (65), on A_n, for $w_i < w_{\text{low}}$,

$$\begin{aligned} \left| \frac{\partial g(w_i;\boldsymbol{\theta})}{\partial \beta} |_{\boldsymbol{\theta}_{*,i}} \right| &< -\left(1/\Phi^{-1}(1/4)\right) \times |\log(\gamma_{*,i} w_i)| \times (\gamma_{*,i} w_i)^{\beta_{*,i}} / \left(1 - \exp(-(\gamma_{*,i} w_i)^{\beta_{*,i}})\right) \\ &\leq -\left((8/7)/\Phi^{-1}(1/4)\right) \times |\log(\gamma_{*,i} w_i)| \end{aligned} \tag{66}$$

From results (35) and (66) and the (ubiquitous) Cauchy–Schwarz inequality, we know that $n > N$ implies that

$$\text{prob}(T_1 < M_4) > 1 - \epsilon \tag{67}$$

where

$$T_1 \equiv \sum_{w_i < w_{\text{low}}} \left(\frac{\partial g(w_i;\boldsymbol{\theta})}{\partial \beta} |_{\boldsymbol{\theta}_{*,i}} \right)^2 / n$$

and

$$M_4 = \left((8/7)/\Phi^{-1}(1/4)\right)^2 \times \left[J^2 + 2J \left(E\left((\log(W))^2 \right) + 1 \right)^{1/2} + E\left((\log(W))^2 \right) + 1 \right]$$

where

$$J \equiv \max\left(|\log(\gamma_{\text{low}})|, |\log(\gamma_{\text{up}})| \right)$$

That is,

$$T_1 = O_p(1) \tag{68}$$

and result (33) follows.

As noted above, results (32) and (33) establish results (30) and (31), which establish result (27) which establishes

$$\sqrt{n}\, D_1 = O_p(1) \tag{69}$$

To complete the proof of the Theorem we now need to show that

$$\sqrt{n}\, D_2 = O_p(1) \tag{70}$$

To establish (70), we first need to establish a few facts about y_i and $\hat{y}_i$. By the Cauchy–Schwarz inequality and result (27), we have

$$\begin{aligned} \sqrt{n} \left| \bar{y} - \hat{\bar{y}} \right| &\leq \sqrt{n} \sum_{i=1}^{n} |y_i - \hat{y}_i| / n \\ &\leq \sqrt{n} \left(\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 / n \right)^{1/2} \\ &= \left(\sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \right)^{1/2} = O_p(1) \end{aligned}$$

Thus,

$$\sqrt{n} \left(\hat{\bar{y}}^2 - \bar{y}^2 \right) = \sqrt{n} \left(\hat{\bar{y}} - \bar{y} \right) \left(\hat{\bar{y}} + \bar{y} \right) = O_p(1) \tag{71}$$

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\sum_{i=1}^{n}(\hat{y}_i + y_i)^2/n &= \sum_{i=1}^{n}(\hat{y}_i - y_i + 2y_i)^2/n \qquad (72)\\ &= \left|\sum_{i=1}^{n}(\hat{y}_i - y_i)^2/n + 4\sum_{i=1}^{n}(\hat{y}_i - y_i)y_i/n + 4\sum_{i=1}^{n} y_i^2/n\right|\\ &\leq \sum_{i=1}^{n}(\hat{y}_i - y_i)^2/n + 4\left(\sum_{i=1}^{n}(\hat{y}_i - y_i)^2/n\right)^{1/2}\left(\sum_{i=1}^{n} y_i^2/n\right)^{1/2} + 4\sum_{i=1}^{n} y_i^2/n\end{aligned}$$

By results (27) and (72) and the fact that

$$\sum_{i=1}^{n} y_i^2/n \xrightarrow{p} E(Y^2)$$

we have

$$\sum_{i=1}^{n}(\hat{y}_i + y_i)^2/n = O_p(1) \qquad (73)$$

By the Cauchy–Schwarz inequality and results (27) and (73) we have

$$\begin{aligned}\sqrt{n}\left|\sum_{i=1}^{n}\left(\hat{y}_i^2 - y_i^2\right)/n\right| &= \sqrt{n}\left|\sum_{i=1}^{n}\left(\hat{y}_i - y_i\right)\left(\hat{y}_i + y_i\right)/n\right| \qquad (74)\\ &\leq \sqrt{n}\left(\sum_{i=1}^{n}\left(\hat{y}_i - y_i\right)^2/n\right)^{1/2}\left(\sum_{i=1}^{n}\left(\hat{y}_i + y_i\right)^2/n\right)^{1/2}\\ &= O_p(1)\end{aligned}$$

By results (71) and (74) we have

$$\begin{aligned}\sqrt{n}\left(\hat{s}_{yy}/n - s_{yy}/n\right) &= \sqrt{n}\left(\sum_{i=1}^{n}\hat{y}_i^2/n - \bar{\hat{y}}^2 - \left(\sum_{i=1}^{n} y_i^2/n - \bar{y}^2\right)\right) \qquad (75)\\ &= \sqrt{n}\left(\sum_{i=1}^{n}\left(\hat{y}_i^2 - y_i^2\right)/n - \left(\bar{\hat{y}}^2 - \bar{y}^2\right)\right)\\ &= O_p(1)\end{aligned}$$

From result (75) we have

$$\sqrt{n}\left(\sqrt{\hat{s}_{yy}/n} - \sqrt{s_{yy}/n}\right) = \sqrt{n}\left(\hat{s}_{yy}/n - s_{yy}/n\right)/\left(\sqrt{\hat{s}_{yy}/n} + \sqrt{s_{yy}/n}\right) = O_p(1) \qquad (76)$$

Now

$$\begin{aligned}D_2 &\equiv \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\hat{y}_i - \bar{\hat{y}})}{\sqrt{s_{xx} \times s_{yy}}} - \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\hat{y}_i - \bar{\hat{y}})}{\sqrt{s_{xx} \times \hat{s}_{yy}}} \qquad (77)\\ &= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(\hat{y}_i - \bar{\hat{y}})}{n} \times \frac{\sqrt{s_{xx} \times \hat{s}_{yy}/n^2} - \sqrt{s_{xx} \times s_{yy}/n^2}}{\sqrt{s_{xx} \times s_{yy} \times s_{xx} \times \hat{s}_{yy}/n^4}}\\ &\equiv F_1 \times F_2\end{aligned}$$

By the Cauchy–Schwarz inequality and (75)

$$
\begin{aligned}
|F_1| &\leq \left(\sum_{i=1}^{n}(x_i-\bar{x})^2/n\right)^{1/2}\left(\sum_{i=1}^{n}(\hat{y}_i-\hat{\bar{y}})^2/n\right)^{1/2} \\
&= \sqrt{s_{xx}/n}\times\sqrt{\hat{s}_{yy}/n}\xrightarrow{p}\sigma\times 1
\end{aligned}
\tag{78}
$$

By results (76) and (77)

$$
\sqrt{n}F_2=\frac{\sqrt{s_{xx}/n}}{\sqrt{s_{xx}\times s_{yy}\times s_{xx}\times\hat{s}_{yy}/n^4}}\times\sqrt{n}\left(\sqrt{\hat{s}_{yy}/n}-\sqrt{s_{yy}/n}\right)=O_p(1) \tag{79}
$$

Results (77), (78), and (79) imply that

$$
\sqrt{n}\,D_2=O_p(1) \tag{80}
$$

This completes the proof of the Theorem. ■

10 Appendix C—Partial Derivatives of $\log(f(x,w))$

Given result (10) we have

$$
\begin{aligned}
\log(f(x,w)) &= -\log(2\pi)-\log(\sigma)-\log\left(\sqrt{1-\rho^2}\right) \\
&\quad -\frac{\left(\frac{x-\mu}{\sigma}\right)^2}{2(1-\rho^2)}+\frac{2\rho\left(\frac{x-\mu}{\sigma}\right)y}{2(1-\rho^2)}-\frac{y^2}{2(1-\rho^2)} \\
&\quad +\beta\log(\gamma)+\log(\beta)+(\beta-1)\log(w) \\
&\quad -(\gamma w)^\beta+\log\left(\sqrt{2\pi}\right)+y^2/2
\end{aligned}
\tag{81}
$$

Thus, [first partials]

$$
\frac{\partial\log(f(x,w))}{\partial\mu}=\frac{x-\mu}{\sigma^2(1-\rho^2)}-\frac{\rho y}{\sigma(1-\rho^2)} \tag{82}
$$

$$
\frac{\partial\log(f(x,w))}{\partial\sigma}=\frac{-1}{\sigma}+\frac{(x-\mu)^2}{\sigma^3(1-\rho^2)}-\frac{\rho(x-\mu)y}{\sigma^2(1-\rho^2)} \tag{83}
$$

$$
\frac{\partial\log(f(x,w))}{\partial\rho}=\frac{\rho}{1-\rho^2}-\left(\left(\frac{x-\mu}{\sigma}\right)^2+y^2\right)\frac{\rho}{(1-\rho^2)^2}+\left(\frac{x-\mu}{\sigma}\right)y\left(\frac{1+\rho^2}{(1-\rho^2)^2}\right) \tag{84}
$$

$$
\frac{\partial\log(f(x,w))}{\partial\gamma}=\frac{\rho\left(\frac{x-\mu}{\sigma}\right)\frac{\partial y}{\partial\gamma}}{1-\rho^2}-\frac{y\frac{\partial y}{\partial\gamma}}{1-\rho^2}+\frac{\beta}{\gamma}-w^\beta\beta\gamma^{\beta-1}+y\frac{\partial y}{\partial\gamma} \tag{85}
$$

$$
\frac{\partial\log(f(x,w))}{\partial\beta}=\frac{\rho\left(\frac{x-\mu}{\sigma}\right)\frac{\partial y}{\partial\beta}}{1-\rho^2}-\frac{y\frac{\partial y}{\partial\beta}}{1-\rho^2}+\log(\gamma)+\frac{1}{\beta}+\log(w)-(\gamma w)^\beta\log(\gamma w)+y\frac{\partial y}{\partial\beta} \tag{86}
$$

and [second main partials]

$$\frac{\partial^2 \log(f(x,w))}{\partial \mu^2} = \frac{-1}{\sigma^2(1-\rho^2)} \tag{87}$$

$$\frac{\partial^2 \log(f(x,w))}{\partial \sigma^2} = \frac{1}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4(1-\rho^2)} + \frac{2\rho(x-\mu)y}{\sigma^3(1-\rho^2)} \tag{88}$$

$$\begin{aligned} \frac{\partial^2 \log(f(x,w))}{\partial \rho^2} &= \frac{1}{1-\rho^2} + \frac{2\rho^2}{(1-\rho^2)^2} - \left(\left(\frac{x-\mu}{\sigma}\right)^2 + y^2\right)\left(\frac{1}{(1-\rho^2)^2} + \frac{4\rho^2}{(1-\rho^2)^3}\right) \\ &\quad + \left(\frac{x-\mu}{\sigma}\right) y \left(\frac{2\rho}{(1-\rho^2)^2} + \frac{4(1+\rho^2)\rho}{(1-\rho^2)^3}\right) \end{aligned} \tag{89}$$

$$\begin{aligned} \frac{\partial^2 \log(f(x,w))}{\partial \gamma^2} &= \frac{\rho\left(\frac{x-\mu}{\sigma}\right)}{1-\rho^2}\frac{\partial^2 y}{\partial \gamma^2} - \frac{1}{1-\rho^2}\left(\left(\frac{\partial y}{\partial \gamma}\right)^2 + y\frac{\partial^2 y}{\partial \gamma^2}\right) \\ &\quad - \frac{\beta}{\gamma^2} - w^\beta \beta(\beta-1)\gamma^{\beta-2} + \left(\left(\frac{\partial y}{\partial \gamma}\right)^2 + y\frac{\partial^2 y}{\partial \gamma^2}\right) \end{aligned} \tag{90}$$

$$\begin{aligned} \frac{\partial^2 \log(f(x,w))}{\partial \beta^2} &= \frac{\rho\left(\frac{x-\mu}{\sigma}\right)}{1-\rho^2}\frac{\partial^2 y}{\partial \beta^2} - \frac{1}{1-\rho^2}\left(\left(\frac{\partial y}{\partial \beta}\right)^2 + y\frac{\partial^2 y}{\partial \beta^2}\right) \\ &\quad - \frac{1}{\beta^2} - (\gamma w)^\beta(\log(\gamma w))^2 + \left(\left(\frac{\partial y}{\partial \beta}\right)^2 + y\frac{\partial^2 y}{\partial \beta^2}\right) \end{aligned} \tag{91}$$

and [second mixed partials, μ]

$$\frac{\partial^2 \log(f(x,w))}{\partial \mu \partial \sigma} = \frac{-2(x-\mu)}{\sigma^3(1-\rho^2)} + \frac{\rho y}{1-\rho^2}\frac{1}{\sigma^2} \tag{92}$$

$$\frac{\partial^2 \log(f(x,w))}{\partial \mu \partial \rho} = \frac{2(x-\mu)}{\sigma^2}\frac{\rho}{(1-\rho^2)^2} - \frac{y(1+\rho^2)}{\sigma(1-\rho^2)^2} \tag{93}$$

$$\frac{\partial^2 \log(f(x,w))}{\partial \mu \partial \gamma} = \frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \gamma}\left(\frac{-1}{\sigma}\right) \tag{94}$$

$$\frac{\partial^2 \log(f(x,w))}{\partial \mu \partial \beta} = \frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \beta}\left(\frac{-1}{\sigma}\right) \tag{95}$$

and [second mixed partials, σ]

$$\frac{\partial^2 \log(f(x,w))}{\partial \sigma \partial \rho} = \frac{2(x-\mu)^2}{\sigma^3}\frac{\rho}{(1-\rho^2)^2} - \frac{(x-\mu)y}{\sigma^2}\frac{(1+\rho^2)}{(1-\rho^2)^2} \tag{96}$$

$$\frac{\partial^2 \log(f(x,w))}{\partial \sigma \partial \gamma} = \frac{-(x-\mu)}{\sigma^2} \times \frac{\rho}{1-\rho^2} \times \frac{\partial y}{\partial \gamma} \tag{97}$$

$$\frac{\partial^2 \log(f(x,w))}{\partial \sigma \partial \beta} = \frac{-(x-\mu)}{\sigma^2} \times \frac{\rho}{1-\rho^2} \times \frac{\partial y}{\partial \beta} \tag{98}$$

and [second mixed partials, ρ]

$$\frac{\partial^2 \log(f(x,w))}{\partial\rho\partial\gamma} = \left(\frac{x-\mu}{\sigma}\right)\frac{\partial y}{\partial\gamma}\left(\frac{1}{1-\rho^2}+\frac{2\rho^2}{(1-\rho^2)^2}\right) - y\times\frac{\partial y}{\partial\gamma}\times\frac{2\rho}{(1-\rho^2)^2} \tag{99}$$

$$\frac{\partial^2 \log(f(x,w))}{\partial\rho\partial\beta} = \left(\frac{x-\mu}{\sigma}\right)\frac{\partial y}{\partial\beta}\left(\frac{1}{1-\rho^2}+\frac{2\rho^2}{(1-\rho^2)^2}\right) - y\times\frac{\partial y}{\partial\beta}\times\frac{2\rho}{(1-\rho^2)^2} \tag{100}$$

and [second mixed partials, γ]

$$\begin{aligned}\frac{\partial^2 \log(f(x,w))}{\partial\gamma\partial\beta} &= \left(\frac{x-\mu}{\sigma}\right)\frac{\rho}{1-\rho^2}\frac{\partial^2 y}{\partial\gamma\partial\beta} - \frac{1}{1-\rho^2}\left(\frac{\partial y}{\partial\gamma}\frac{\partial y}{\partial\beta}+y\frac{\partial^2 y}{\partial\gamma\partial\beta}\right) \\ &\quad + 1/\gamma - \left(w^\beta\log(w)\beta\gamma^{\beta-1}+w^\beta\gamma^{\beta-1}+w^\beta\beta\gamma^{\beta-1}\log(\gamma)\right) \\ &\quad + \frac{\partial y}{\partial\gamma}\frac{\partial y}{\partial\beta}+y\frac{\partial^2 y}{\partial\gamma\partial\beta}\end{aligned} \tag{101}$$

and [third main partials]

1:

$$\frac{\partial^3 \log(f(x,w))}{\partial\mu^3} = 0 \tag{102}$$

2:

$$\frac{\partial^3 \log(f(x,w))}{\partial\sigma^3} = \frac{-2}{\sigma^3}+\frac{12(x-\mu)^2}{\sigma^5(1-\rho^2)}-\frac{6\rho(x-\mu)y}{\sigma^4(1-\rho^2)} \tag{103}$$

3:

$$\begin{aligned}\frac{\partial^3 \log(f(x,w))}{\partial\rho^3} &= \frac{6\rho}{(1-\rho^2)^2}+\frac{8\rho^3}{(1-\rho^2)^3} \\ &\quad - \left(\left(\frac{x-\mu}{\sigma}\right)^2+y^2\right)\left(\frac{12\rho}{(1-\rho^2)^3}+\frac{24\rho^3}{(1-\rho^2)^4}\right) \\ &\quad + \left(\frac{x-\mu}{\sigma}\right)y\left(\frac{2}{(1-\rho^2)^2}+\frac{8\rho^2}{(1-\rho^2)^3}+\frac{4(1+3\rho^2)}{(1-\rho^2)^3}+\frac{24\rho^2(1+\rho^2)}{(1-\rho^2)^4}\right)\end{aligned} \tag{104}$$

4:

$$\begin{aligned}\frac{\partial^3 \log(f(x,w))}{\partial\gamma^3} &= \frac{\rho}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\frac{\partial^3 y}{\partial\gamma^3} - \frac{1}{1-\rho^2}\left(3\frac{\partial y}{\partial\gamma}\frac{\partial^2 y}{\partial\gamma^2}+y\frac{\partial^3 y}{\partial\gamma^3}\right) \\ &\quad + \frac{2\beta}{\gamma^3} - w^\beta\beta(\beta-1)(\beta-2)\gamma^{\beta-3} + \left(3\frac{\partial y}{\partial\gamma}\frac{\partial^2 y}{\partial\gamma^2}+y\frac{\partial^3 y}{\partial\gamma^3}\right)\end{aligned} \tag{105}$$

5:

$$\begin{aligned}\frac{\partial^3 \log(f(x,w))}{\partial\beta^3} &= \frac{\rho}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\frac{\partial^3 y}{\partial\beta^3} - \frac{1}{1-\rho^2}\left(3\frac{\partial y}{\partial\beta}\frac{\partial^2 y}{\partial\beta^2}+y\frac{\partial^3 y}{\partial\beta^3}\right) \\ &\quad + \frac{2}{\beta^3} - (\gamma w)^\beta\left(\log(\gamma w)\right)^3 + \left(3\frac{\partial y}{\partial\beta}\frac{\partial^2 y}{\partial\beta^2}+y\frac{\partial^3 y}{\partial\beta^3}\right)\end{aligned} \tag{106}$$

and [third partials, 2 with respect to μ]

6:

$$\frac{\partial^3 \log(f(x,w))}{\partial \mu^2 \partial \sigma} = \frac{2}{\sigma^3(1-\rho^2)} \tag{107}$$

7:

$$\frac{\partial^3 \log(f(x,w))}{\partial \mu^2 \partial \rho} = \frac{-2\rho}{\sigma^2(1-\rho^2)^2} \tag{108}$$

8:

$$\frac{\partial^3 \log(f(x,w))}{\partial \mu^2 \partial \gamma} = 0 \tag{109}$$

9:

$$\frac{\partial^3 \log(f(x,w))}{\partial \mu^2 \partial \beta} = 0 \tag{110}$$

and [third partials, 2 with respect to σ]

10:

$$\frac{\partial^3 \log(f(x,w))}{\partial \sigma^2 \partial \mu} = \frac{6(x-\mu)}{\sigma^4(1-\rho^2)} - \frac{2y\rho}{\sigma^3(1-\rho^2)} \tag{111}$$

11:

$$\frac{\partial^3 \log(f(x,w))}{\partial \sigma^2 \partial \rho} = \frac{-6(x-\mu)^2}{\sigma^4}\frac{\rho}{(1-\rho^2)^2} + \frac{2(x-\mu)y}{\sigma^3}\frac{1+\rho^2}{(1-\rho^2)^2} \tag{112}$$

12:

$$\frac{\partial^3 \log(f(x,w))}{\partial \sigma^2 \partial \gamma} = \frac{2(x-\mu)}{\sigma^3} \times \frac{\rho}{1-\rho^2} \times \frac{\partial y}{\partial \gamma} \tag{113}$$

13:

$$\frac{\partial^3 \log(f(x,w))}{\partial \sigma^2 \partial \beta} = \frac{2(x-\mu)}{\sigma^3} \times \frac{\rho}{1-\rho^2} \times \frac{\partial y}{\partial \beta} \tag{114}$$

and [third partials, 2 with respect to ρ]

14:

$$\frac{\partial^3 \log(f(x,w))}{\partial \rho^2 \partial \mu} = \frac{2(x-\mu)}{\sigma^2}\left(\frac{1}{(1-\rho^2)^2} + \frac{4\rho^2}{(1-\rho^2)^3}\right) - \frac{y}{\sigma}\left(\frac{2\rho}{(1-\rho^2)^2} + \frac{4\rho(1+\rho^2)}{(1-\rho^2)^3}\right) \tag{115}$$

15:

$$\begin{aligned}\frac{\partial^3 \log(f(x,w))}{\partial \rho^2 \partial \sigma} &= \frac{2(x-\mu)^2}{\sigma^3}\left(\frac{1}{(1-\rho^2)^2} + \frac{4\rho^2}{(1-\rho^2)^3}\right) \\ &\quad - \frac{(x-\mu)y}{\sigma^2}\left(\frac{2\rho}{(1-\rho^2)^2} + \frac{4\rho(1+\rho^2)}{(1-\rho^2)^3}\right)\end{aligned} \tag{116}$$

16:

$$\begin{aligned}\frac{\partial^3 \log(f(x,w))}{\partial \rho^2 \partial \gamma} &= \left(\frac{x-\mu}{\sigma}\right)\frac{\partial y}{\partial \gamma}\left(\frac{2\rho}{(1-\rho^2)^2} + \frac{4\rho}{(1-\rho^2)^2} + \frac{8\rho^3}{(1-\rho^2)^3}\right) \\ &\quad - y\frac{\partial y}{\partial \gamma}\left(\frac{2}{(1-\rho^2)^2} + \frac{8\rho^2}{(1-\rho^2)^3}\right)\end{aligned} \tag{117}$$

17:

$$\begin{aligned}\frac{\partial^3 \log(f(x,w))}{\partial \rho^2 \partial \beta} &= \left(\frac{x-\mu}{\sigma}\right)\frac{\partial y}{\partial \beta}\left(\frac{2\rho}{(1-\rho^2)^2} + \frac{4\rho}{(1-\rho^2)^2} + \frac{8\rho^3}{(1-\rho^2)^3}\right) \\ &\quad - y\frac{\partial y}{\partial \beta}\left(\frac{2}{(1-\rho^2)^2} + \frac{8\rho^2}{(1-\rho^2)^3}\right)\end{aligned} \tag{118}$$

and [third partials, 2 with respect to γ]

18:

$$\frac{\partial^3 \log(f(x,w))}{\partial\gamma^2\partial\mu} = \frac{-1}{\sigma} \times \frac{\rho}{1-\rho^2} \times \frac{\partial^2 y}{\partial\gamma^2} \tag{119}$$

19:

$$\frac{\partial^3 \log(f(x,w))}{\partial\gamma^2\partial\sigma} = \frac{-(x-\mu)}{\sigma^2} \times \frac{\rho}{1-\rho^2} \times \frac{\partial^2 y}{\partial\gamma^2} \tag{120}$$

20:

$$\frac{\partial^3 \log(f(x,w))}{\partial\gamma^2\partial\rho} = \left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{1-\rho^2} + \frac{2\rho^2}{(1-\rho^2)^2}\right)\frac{\partial^2 y}{\partial\gamma^2} - \frac{2\rho}{(1-\rho^2)^2}\left(\left(\frac{\partial y}{\partial\gamma}\right)^2 + y\frac{\partial^2 y}{\partial\gamma^2}\right) \tag{121}$$

21:

$$\begin{aligned}\frac{\partial^3 \log(f(x,w))}{\partial\gamma^2\partial\beta} \quad = \quad & \left(\frac{x-\mu}{\sigma}\right) \times \frac{\rho}{1-\rho^2} \times \frac{\partial^3 y}{\partial\gamma^2\partial\beta} \\ & - \frac{1}{1-\rho^2}\left(2 \times \frac{\partial y}{\partial\gamma} \times \frac{\partial^2 y}{\partial\gamma\partial\beta} + \frac{\partial y}{\partial\beta} \times \frac{\partial^2 y}{\partial\gamma^2} + y \times \frac{\partial^3 y}{\partial\gamma^2\partial\beta}\right) \\ & - \frac{1}{\gamma^2} - \left(w^\beta \log(w)(\beta^2-\beta)\gamma^{\beta-2} + w^\beta(2\beta-1)\gamma^{\beta-2} + w^\beta(\beta^2-\beta)\gamma^{\beta-2}\log(\gamma)\right) \\ & + \left(2 \times \frac{\partial y}{\partial\gamma} \times \frac{\partial^2 y}{\partial\gamma\partial\beta} + \frac{\partial y}{\partial\beta} \times \frac{\partial^2 y}{\partial\gamma^2} + y \times \frac{\partial^3 y}{\partial\gamma^2\partial\beta}\right)\end{aligned} \tag{122}$$

and [third partials, 2 with respect to β]

22:

$$\frac{\partial^3 \log(f(x,w))}{\partial\beta^2\partial\mu} = \frac{-1}{\sigma} \times \frac{\rho}{1-\rho^2} \times \frac{\partial^2 y}{\partial\beta^2} \tag{123}$$

23:

$$\frac{\partial^3 \log(f(x,w))}{\partial\beta^2\partial\sigma} = \frac{-(x-\mu)}{\sigma^2} \times \frac{\rho}{1-\rho^2} \times \frac{\partial^2 y}{\partial\beta^2} \tag{124}$$

24:

$$\frac{\partial^3 \log(f(x,w))}{\partial\beta^2\partial\rho} = \left(\frac{x-\mu}{\sigma}\right)\left(\frac{1}{1-\rho^2} + \frac{2\rho^2}{(1-\rho^2)^2}\right)\frac{\partial^2 y}{\partial\beta^2} - \frac{2\rho}{(1-\rho^2)^2}\left(\left(\frac{\partial y}{\partial\beta}\right)^2 + y\frac{\partial^2 y}{\partial\beta^2}\right) \tag{125}$$

25:

$$\begin{aligned}\frac{\partial^3 \log(f(x,w))}{\partial\beta^2\partial\gamma} \quad = \quad & \left(\frac{x-\mu}{\sigma}\right) \times \frac{\rho}{1-\rho^2} \times \frac{\partial^3 y}{\partial\gamma\partial\beta^2} \\ & - \frac{1}{1-\rho^2}\left(2 \times \frac{\partial y}{\partial\beta} \times \frac{\partial^2 y}{\partial\gamma\partial\beta} + \frac{\partial y}{\partial\gamma} \times \frac{\partial^2 y}{\partial\beta^2} + y \times \frac{\partial^3 y}{\partial\gamma\partial\beta^2}\right) \\ & - \left(w^\beta \beta\gamma^{\beta-1}(\log(\gamma w))^2 + (\gamma w)^\beta 2\log(\gamma w)/\gamma\right) \\ & + \left(2 \times \frac{\partial y}{\partial\beta} \times \frac{\partial^2 y}{\partial\gamma\partial\beta} + \frac{\partial y}{\partial\gamma} \times \frac{\partial^2 y}{\partial\beta^2} + y \times \frac{\partial^3 y}{\partial\gamma\partial\beta^2}\right)\end{aligned} \tag{126}$$

and [third partials, μ, σ, ρ]

26:

$$\frac{\partial^3 \log(f(x,w))}{\partial\mu\partial\sigma\partial\rho} = \frac{-4(x-\mu)}{\sigma^3} \times \frac{\rho}{(1-\rho^2)^2} + \frac{y}{\sigma^2} \times \left(\frac{1}{1-\rho^2} + \frac{2\rho^2}{(1-\rho^2)^2}\right) \tag{127}$$

and [third partials, μ, σ, γ]
27:

$$\frac{\partial^3 \log(f(x,w))}{\partial\mu\partial\sigma\partial\gamma} = \frac{1}{\sigma^2} \times \frac{\rho}{1-\rho^2} \times \frac{\partial y}{\partial\gamma} \tag{128}$$

and [third partials, μ, σ, β]
28:

$$\frac{\partial^3 \log(f(x,w))}{\partial\mu\partial\sigma\partial\beta} = \frac{1}{\sigma^2} \times \frac{\rho}{1-\rho^2} \times \frac{\partial y}{\partial\beta} \tag{129}$$

and [third partials, μ, ρ, γ]
29:

$$\frac{\partial^3 \log(f(x,w))}{\partial\mu\partial\rho\partial\gamma} = \frac{-1}{\sigma} \times \frac{1+\rho^2}{(1-\rho^2)^2} \times \frac{\partial y}{\partial\gamma} \tag{130}$$

and [third partials, μ, ρ, β]
30:

$$\frac{\partial^3 \log(f(x,w))}{\partial\mu\partial\rho\partial\beta} = \frac{-1}{\sigma} \times \frac{1+\rho^2}{(1-\rho^2)^2} \times \frac{\partial y}{\partial\beta} \tag{131}$$

and [third partials, μ, γ, β]
31:

$$\frac{\partial^3 \log(f(x,w))}{\partial\mu\partial\gamma\partial\beta} = \frac{-1}{\sigma} \times \frac{\rho}{1-\rho^2} \times \frac{\partial^2 y}{\partial\gamma\partial\beta} \tag{132}$$

and [third partials, σ, ρ, γ]
32:

$$\frac{\partial^3 \log(f(x,w))}{\partial\sigma\partial\rho\partial\gamma} = \frac{-(x-\mu)}{\sigma^2} \times \frac{1+\rho^2}{(1-\rho^2)^2} \times \frac{\partial y}{\partial\gamma} \tag{133}$$

and [third partials, σ, ρ, β]
33:

$$\frac{\partial^3 \log(f(x,w))}{\partial\sigma\partial\rho\partial\beta} = \frac{-(x-\mu)}{\sigma^2} \times \frac{1+\rho^2}{(1-\rho^2)^2} \times \frac{\partial y}{\partial\beta} \tag{134}$$

and [third partials, σ, γ, β]
34:

$$\frac{\partial^3 \log(f(x,w))}{\partial\sigma\partial\gamma\partial\beta} = \frac{-(x-\mu)}{\sigma^2} \times \frac{\rho}{1-\rho^2} \times \frac{\partial^2 y}{\partial\gamma\partial\beta} \tag{135}$$

and [third partials, ρ, γ, β]
35:

$$\frac{\partial^3 \log(f(x,w))}{\partial\rho\partial\gamma\partial\beta} = \frac{x-\mu}{\sigma}\left(\frac{1}{1-\rho^2} + \frac{2\rho^2}{(1-\rho^2)^2}\right)\frac{\partial^2 y}{\partial\gamma\partial\beta} - \frac{2\rho}{(1-\rho^2)^2}\left(\frac{\partial y}{\partial\gamma}\frac{\partial y}{\partial\beta} + y\frac{\partial^2 y}{\partial\gamma\partial\beta}\right) \tag{136}$$

11 Appendix D—Partial Derivatives of y

Recall that

$$y \equiv \Phi^{-1}\left(1 - \exp(-(\gamma w)^\beta\right)$$

Thus, we have

$$\begin{aligned}\frac{\partial y}{\partial\gamma} &= \frac{\exp\left(-(\gamma w)^\beta\right)\beta\gamma^{\beta-1}w^\beta}{\phi(y)} \\ &= \sqrt{2\pi} \times \beta\gamma^{\beta-1} \times w^\beta \times \exp\left(-(\gamma w)^\beta\right) \times \exp\left(y^2/2\right)\end{aligned} \tag{137}$$

$$\frac{\partial y}{\partial \beta} = \sqrt{2\pi} \times (\gamma w)^{\beta} \log(\gamma w) \times \exp\left(-(\gamma w)^{\beta}\right) \times \exp\left(y^2/2\right) \tag{138}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial \gamma^2} &= \sqrt{2\pi}\, \beta w^{\beta} \Big[(\beta - 1)\gamma^{\beta-2} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) \\ &\quad + \gamma^{\beta-1} \exp\left(-(\gamma w)^{\beta}\right) (-1) w^{\beta} \beta \gamma^{\beta-1} \exp\left(y^2/2\right) \\ &\quad + \gamma^{\beta-1} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) y \frac{\partial y}{\partial \gamma}\Big] \end{aligned} \tag{139}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial \beta^2} &= \sqrt{2\pi}\, \log(\gamma w) \Big[(\gamma w)^{\beta} \log(\gamma w) \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) \\ &\quad + (\gamma w)^{\beta} \exp\left(-(\gamma w)^{\beta}\right) (-1)(\gamma w)^{\beta} \log(\gamma w) \exp\left(y^2/2\right) \\ &\quad + (\gamma w)^{\beta} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) y \frac{\partial y}{\partial \beta}\Big] \end{aligned} \tag{140}$$

$$\begin{aligned} \frac{\partial^2 y}{\partial \gamma \partial \beta} &= \sqrt{2\pi} \Big[(1/\gamma)(\gamma w)^{\beta} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) \\ &\quad + \log(\gamma w) w^{\beta} \beta \gamma^{\beta-1} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) \\ &\quad + \log(\gamma w)(\gamma w)^{\beta} \exp\left(-(\gamma w)^{\beta}\right) (-1) w^{\beta} \beta \gamma^{\beta-1} \exp\left(y^2/2\right) \\ &\quad + \log(\gamma w)(\gamma w)^{\beta} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) y \frac{\partial y}{\partial \gamma}\Big] \end{aligned} \tag{141}$$

$$\frac{\partial^3 y}{\partial \gamma^3} = \sqrt{2\pi} \beta w^{\beta} \left((\beta - 1) \times T_1 - w^{\beta} \beta \times T_2 + T_3\right) \tag{142}$$

where

$$\begin{aligned} T_1 &= (\beta - 2)\gamma^{\beta-3} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) \\ &\quad + \gamma^{\beta-2} \exp\left(-(\gamma w)^{\beta}\right) (-1) w^{\beta} \beta \gamma^{\beta-1} \exp\left(y^2/2\right) \\ &\quad + \gamma^{\beta-2} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) y \frac{\partial y}{\partial \gamma} \end{aligned}$$

$$\begin{aligned} T_2 &= (2\beta - 2)\gamma^{2\beta-3} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) \\ &\quad + \gamma^{2\beta-2} \exp\left(-(\gamma w)^{\beta}\right) (-1) w^{\beta} \beta \gamma^{\beta-1} \exp\left(y^2/2\right) \\ &\quad + \gamma^{2\beta-2} \exp\left(-(\gamma w)^{\beta}\right) \exp\left(y^2/2\right) y \frac{\partial y}{\partial \gamma} \end{aligned}$$

$$
\begin{aligned}
T_3 &= (\beta-1)\gamma^{\beta-2}\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y\frac{\partial y}{\partial\gamma} \\
&\quad+\gamma^{\beta-1}\exp\left(-(\gamma w)^\beta\right)(-1)w^\beta\beta\gamma^{\beta-1}\exp\left(y^2/2\right)y\frac{\partial y}{\partial\gamma} \\
&\quad+\gamma^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y^2\left(\frac{\partial y}{\partial\gamma}\right)^2 \\
&\quad+\gamma^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)\left(\frac{\partial y}{\partial\gamma}\right)^2 \\
&\quad+\gamma^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y\frac{\partial^2 y}{\partial\gamma^2}
\end{aligned}
$$

$$
\frac{\partial^3 y}{\partial\beta^3}=\sqrt{2\pi}\log(\gamma w)\left(\log(\gamma w)\times T_1-\log(\gamma w)\times T_2+T_3\right) \tag{143}
$$

where

$$
\begin{aligned}
T_1 &= (\gamma w)^\beta\log(\gamma w)\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right) \\
&\quad+(\gamma w)^\beta\exp\left(-(\gamma w)^\beta\right)(-1)(\gamma w)^\beta\log(\gamma w)\exp\left(y^2/2\right) \\
&\quad+(\gamma w)^\beta\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y\frac{\partial y}{\partial\beta}
\end{aligned}
$$

$$
\begin{aligned}
T_2 &= (\gamma w)^{2\beta}2\log(\gamma w)\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right) \\
&\quad+(\gamma w)^{2\beta}\exp\left(-(\gamma w)^\beta\right)(-1)(\gamma w)^\beta\log(\gamma w)\exp\left(y^2/2\right) \\
&\quad+(\gamma w)^{2\beta}\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y\frac{\partial y}{\partial\beta}
\end{aligned}
$$

$$
\begin{aligned}
T_3 &= (\gamma w)^\beta\log(\gamma w)\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y\frac{\partial y}{\partial\beta} \\
&\quad+(\gamma w)^\beta\exp\left(-(\gamma w)^\beta\right)(-1)(\gamma w)^\beta\log(\gamma w)\exp\left(y^2/2\right)y\frac{\partial y}{\partial\beta} \\
&\quad+(\gamma w)^\beta\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y^2\left(\frac{\partial y}{\partial\beta}\right)^2 \\
&\quad+(\gamma w)^\beta\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)\left(\frac{\partial y}{\partial\beta}\right)^2 \\
&\quad+(\gamma w)^\beta\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y\frac{\partial^2 y}{\partial\beta^2}
\end{aligned}
$$

$$
\frac{\partial^3 y}{\partial\gamma^2\partial\beta}=\sqrt{2\pi}\left(w^\beta\times T_1+w^\beta\beta\times T_2-w^\beta\beta\times T_3+w^\beta\times T_4\right) \tag{144}
$$

where

$$
\begin{aligned}
T_1 &= (\beta-1)\gamma^{\beta-2}\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right) \\
&\quad+\gamma^{\beta-1}\exp\left(-(\gamma w)^\beta\right)(-1)w^\beta\beta\gamma^{\beta-1}\exp\left(y^2/2\right) \\
&\quad+\gamma^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\exp\left(y^2/2\right)y\frac{\partial y}{\partial\gamma}
\end{aligned}
$$

$$
\begin{aligned}
T_2 \;=\;& (1/\gamma)\gamma^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) \\
&+ \log(\gamma w)(\beta-1)\gamma^{\beta-2}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) \\
&+ \log(\gamma w)\gamma^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)(-1)w^{\beta}\beta\gamma^{\beta-1}\exp\left(y^2/2\right) \\
&+ \log(\gamma w)\gamma^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) y\frac{\partial y}{\partial \gamma}
\end{aligned}
$$

$$
\begin{aligned}
T_3 \;=\;& (1/\gamma)(\gamma w)^{\beta}\exp\left(-(\gamma w)^{\beta}\right)\gamma^{\beta-1}\exp\left(y^2/2\right) \\
&+ \log(\gamma w)w^{\beta}\beta\gamma^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\gamma^{\beta-1}\exp\left(y^2/2\right) \\
&+ \log(\gamma w)(\gamma w)^{\beta}\exp\left(-(\gamma w)^{\beta}\right)(-1)w^{\beta}\beta\gamma^{\beta-1}\gamma^{\beta-1}\exp\left(y^2/2\right) \\
&+ \log(\gamma w)(\gamma w)^{\beta}\exp\left(-(\gamma w)^{\beta}\right)(\beta-1)\gamma^{\beta-2}\exp\left(y^2/2\right) \\
&+ \log(\gamma w)(\gamma w)^{\beta}\exp\left(-(\gamma w)^{\beta}\right)\gamma^{\beta-1}\exp\left(y^2/2\right) y\frac{\partial y}{\partial \gamma}
\end{aligned}
$$

$$
\begin{aligned}
T_4 \;=\;& (1/\gamma)\gamma^{\beta}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) y\frac{\partial y}{\partial \gamma} \\
&+ \log(\gamma w)\beta\gamma^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) y\frac{\partial y}{\partial \gamma} \\
&+ \log(\gamma w)\gamma^{\beta}\exp\left(-(\gamma w)^{\beta}\right)(-1)w^{\beta}\beta\gamma^{\beta-1}\exp\left(y^2/2\right) y\frac{\partial y}{\partial \gamma} \\
&+ \log(\gamma w)\gamma^{\beta}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) y^2\left(\frac{\partial y}{\partial \gamma}\right)^2 \\
&+ \log(\gamma w)\gamma^{\beta}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right)\left(\frac{\partial y}{\partial \gamma}\right)^2 \\
&+ \log(\gamma w)\gamma^{\beta}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) y\frac{\partial^2 y}{\partial \gamma^2}
\end{aligned}
$$

$$
\frac{\partial^3 y}{\partial\gamma\partial\beta^2} = \sqrt{2\pi}\left(T_1 + \log(\gamma w)\times T_2 - \log(\gamma w)\times T_3 + \log(\gamma w)\times T_4\right) \tag{145}
$$

where

$$
\begin{aligned}
T_1 \;=\;& \gamma^{\beta-1}\log(\gamma)w^{\beta}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) \\
&+ \gamma^{\beta-1}w^{\beta}\log(w)\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) \\
&+ \gamma^{\beta-1}w^{\beta}\exp\left(-(\gamma w)^{\beta}\right)(-1)(\gamma w)^{\beta}\log(\gamma w)\exp\left(y^2/2\right) \\
&+ \gamma^{\beta-1}w^{\beta}\exp\left(-(\gamma w)^{\beta}\right)\exp\left(y^2/2\right) y\frac{\partial y}{\partial \beta}
\end{aligned}
$$

$$
\begin{aligned}
T_2 \quad = \quad & w^\beta \log(w)\beta\gamma^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \exp\left(y^2/2\right) \\
& + w^\beta \gamma^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \exp\left(y^2/2\right) \\
& + w^\beta \beta\gamma^{\beta-1} \log(\gamma) \exp\left(-(\gamma w)^\beta\right) \exp\left(y^2/2\right) \\
& + w^\beta \beta\gamma^{\beta-1} \exp\left(-(\gamma w)^\beta\right) (-1)(\gamma w)^\beta \log(\gamma w) \exp\left(y^2/2\right) \\
& + w^\beta \beta\gamma^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \exp\left(y^2/2\right) y \frac{\partial y}{\partial \beta}
\end{aligned}
$$

$$
\begin{aligned}
T_3 \quad = \quad & (\gamma w)^\beta \log(\gamma w) \exp\left(-(\gamma w)^\beta\right) w^\beta \beta\gamma^{\beta-1} \exp\left(y^2/2\right) \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) (-1)(\gamma w)^\beta \log(\gamma w) w^\beta \beta\gamma^{\beta-1} \exp\left(y^2/2\right) \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) w^\beta \log(w)\beta\gamma^{\beta-1} \exp\left(y^2/2\right) \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) w^\beta \gamma^{\beta-1} \exp\left(y^2/2\right) \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) w^\beta \beta\gamma^{\beta-1} \log(\gamma) \exp\left(y^2/2\right) \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) w^\beta \beta\gamma^{\beta-1} \exp\left(y^2/2\right) y \frac{\partial y}{\partial \beta}
\end{aligned}
$$

$$
\begin{aligned}
T_4 \quad = \quad & (\gamma w)^\beta \log(\gamma w) \exp\left(-(\gamma w)^\beta\right) \exp\left(y^2/2\right) y \frac{\partial y}{\partial \gamma} \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) (-1)(\gamma w)^\beta \log(\gamma w) \exp\left(y^2/2\right) y \frac{\partial y}{\partial \gamma} \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) \exp\left(y^2/2\right) y^2 \frac{\partial y}{\partial \gamma} \frac{\partial y}{\partial \beta} \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) \exp\left(y^2/2\right) \frac{\partial y}{\partial \gamma} \frac{\partial y}{\partial \beta} \\
& + (\gamma w)^\beta \exp\left(-(\gamma w)^\beta\right) \exp\left(y^2/2\right) y \frac{\partial^2 y}{\partial \gamma \partial \beta}
\end{aligned} \tag{146}
$$

12 Appendix E1—Expectations of the first partials of the likelihood

Recall that, by construction, x, y are normal random variables with correlation 1, means μ and 0, respectively, and standard deviations σ and 1, respectively.

Thus, from result (82), we have

$$
E\left(\frac{\partial \log(f(x,w))}{\partial \mu}\right) = 0 - 0 = 0 \tag{147}
$$

From result (83), we have

$$
E\left(\frac{\partial \log(f(x,w))}{\partial \sigma}\right) = \frac{-1}{\sigma} + \frac{\sigma^2}{\sigma^3(1-\rho^2)} - \frac{\rho^2\sigma}{\sigma^2(1-\rho^2)} = \frac{-1}{\sigma} + \frac{1}{\sigma(1-\rho^2)} - \frac{\rho^2}{\sigma(1-\rho^2)} = 0 \tag{148}
$$

From result (84), we have

$$E\left(\frac{\partial \log(f(x,w))}{\partial \rho}\right) = \frac{\rho}{1-\rho^2} - \frac{2\rho}{(1-\rho^2)^2} + \frac{\rho(1+\rho^2)}{(1-\rho^2)^2} = 0 \tag{149}$$

From results (85), (261), and (277), we have

$$E\left(\frac{\partial \log(f(x,w))}{\partial \gamma}\right) = \frac{\rho^2 E\left(y\frac{\partial y}{\partial \gamma}\right)}{1-\rho^2} - \frac{E\left(y\frac{\partial y}{\partial \gamma}\right)}{1-\rho^2} + 0 + E\left(y\frac{\partial y}{\partial \gamma}\right) = 0 \tag{150}$$

From results (86), (262), and (278), we have

$$E\left(\frac{\partial \log(f(x,w))}{\partial \beta}\right) = \frac{\rho^2 E\left(y\frac{\partial y}{\partial \beta}\right)}{1-\rho^2} - \frac{E\left(y\frac{\partial y}{\partial \beta}\right)}{1-\rho^2} + 0 + E\left(y\frac{\partial y}{\partial \beta}\right) = 0 \tag{151}$$

13 Appendix E2—Expectations of the second partials of the likelihood

(The existence and finiteness of the expectations involving partial derivatives of y are established in Appendix H.)

Recall that, by construction, x, y are normal random variables with correlation 1, means μ and 0, respectively, and standard deviations σ and 1, respectively.

Thus, from result (87), we have

$$E\left(\frac{\partial^2 \log(f(x,w))}{\partial \mu^2}\right) = \frac{-1}{\sigma^2(1-\rho^2)} \tag{152}$$

From result (88), we have

$$E\left(\frac{\partial^2 \log(f(x,w))}{\partial \sigma^2}\right) = \frac{1}{\sigma^2} - \frac{3}{\sigma^2(1-\rho^2)} + \frac{2\rho^2}{\sigma^2(1-\rho^2)} = \frac{-2+\rho^2}{\sigma^2(1-\rho^2)} \tag{153}$$

From result (89), we have

$$\begin{aligned} E\left(\frac{\partial^2 \log(f(x,w))}{\partial \rho^2}\right) &= \frac{1}{1-\rho^2} + \frac{2\rho^2}{(1-\rho^2)^2} - 2\left(\frac{1}{(1-\rho^2)^2} + \frac{4\rho^2}{(1-\rho^2)^3}\right) \\ &\quad + \rho\left(\frac{2\rho}{(1-\rho^2)^2} + \frac{4(1+\rho^2)\rho}{(1-\rho^2)^3}\right) \\ &= \frac{-(1+\rho^2)}{(1-\rho^2)^2} \end{aligned} \tag{154}$$

From results (90), (256), and (282), we have

$$\begin{aligned} E\left(\frac{\partial^2 \log(f(x,w))}{\partial \gamma^2}\right) &= \frac{\rho^2}{1-\rho^2} E\left(y\frac{\partial^2 y}{\partial \gamma^2}\right) - \frac{\rho^2}{1-\rho^2} E\left(\left(\frac{\partial y}{\partial \gamma}\right)^2 + y\frac{\partial^2 y}{\partial \gamma^2}\right) \\ &\quad - \frac{\beta^2}{\gamma^2} \\ &= \frac{-\rho^2}{1-\rho^2} E\left(\left(\frac{\partial y}{\partial \gamma}\right)^2\right) - \frac{\beta^2}{\gamma^2} \end{aligned} \tag{155}$$

From results (91), (256), and (287), we have

$$
\begin{aligned}
E\left(\frac{\partial^2 \log(f(x,w))}{\partial\beta^2}\right) &= \frac{\rho^2}{1-\rho^2}E\left(y\frac{\partial^2 y}{\partial\beta^2}\right) - \frac{\rho^2}{1-\rho^2}E\left(\left(\frac{\partial y}{\partial\beta}\right)^2 + y\frac{\partial^2 y}{\partial\beta^2}\right) \\
&\quad - \frac{1}{\beta^2} - E\left((\gamma w)^\beta(\log(\gamma w))^2\right) \\
&= \frac{-\rho^2}{1-\rho^2}E\left(\left(\frac{\partial y}{\partial\beta}\right)^2\right) - \frac{1}{\beta^2} - E\left((\gamma w)^\beta(\log(\gamma w))^2\right) \\
&= \frac{-\rho^2}{1-\rho^2}E\left(\left(\frac{\partial y}{\partial\beta}\right)^2\right) - \frac{1}{\beta^2} \\
&\quad - E\left((\log(w))^2\right) - \frac{2}{\beta}E\left(\log(w)\right) \\
&\quad - 2\log(\gamma)E\left(\log(w)\right) - \frac{2\log(\gamma)}{\beta} \\
&\quad - \left(\log(\gamma)\right)^2
\end{aligned} \tag{156}
$$

From result (92), we have

$$
E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu\partial\sigma}\right) = \frac{-2E(x-\mu)}{\sigma^3(1-\rho^2)} + \frac{\rho E(y)}{1-\rho^2}\frac{1}{\sigma^2} = 0 \tag{157}
$$

From result (93), we have

$$
E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu\partial\rho}\right) = \frac{2E(x-\mu)}{\sigma^2}\frac{\rho}{(1-\rho^2)^2} - \frac{E(y)(1+\rho^2)}{\sigma(1-\rho^2)^2} = 0 \tag{158}
$$

From result (94), we have

$$
E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu\partial\gamma}\right) = \frac{\rho}{1-\rho^2}E\left(\frac{\partial y}{\partial\gamma}\right)\left(\frac{-1}{\sigma}\right) \tag{159}
$$

From result (95), we have

$$
E\left(\frac{\partial^2 \log(f(x,w))}{\partial\mu\partial\beta}\right) = \frac{\rho}{1-\rho^2}E\left(\frac{\partial y}{\partial\beta}\right)\left(\frac{-1}{\sigma}\right) \tag{160}
$$

From results (96) and (256), we have

$$
\begin{aligned}
E\left(\frac{\partial^2 \log(f(x,w))}{\partial\sigma\partial\rho}\right) &= \frac{2\rho}{\sigma(1-\rho^2)^2} - \frac{\rho(1+\rho^2)}{\sigma(1-\rho^2)^2} \\
&= \frac{\rho}{\sigma(1-\rho^2)}
\end{aligned} \tag{161}
$$

From results (97) and (256), we have

$$
E\left(\frac{\partial^2 \log(f(x,w))}{\partial\sigma\partial\gamma}\right) = \frac{-\rho^2}{\sigma(1-\rho^2)}E\left(y\frac{\partial y}{\partial\gamma}\right) \tag{162}
$$

From results (98) and (256), we have

$$
E\left(\frac{\partial^2 \log(f(x,w))}{\partial\sigma\partial\beta}\right) = \frac{-\rho^2}{\sigma(1-\rho^2)}E\left(y\frac{\partial y}{\partial\beta}\right) \tag{163}
$$

From results (99) and (256), we have

$$E\left(\frac{\partial^2 \log(f(x,w))}{\partial\rho\partial\gamma}\right) = \left(\frac{\rho}{1-\rho^2} + \frac{2\rho^3}{(1-\rho^2)^2} - \frac{2\rho}{(1-\rho^2)^2}\right) E\left(y\frac{\partial y}{\partial \gamma}\right) = \frac{-\rho}{1-\rho^2} \times E\left(y\frac{\partial y}{\partial \gamma}\right) \tag{164}$$

From results (100) and (256), we have

$$E\left(\frac{\partial^2 \log(f(x,w))}{\partial\rho\partial\beta}\right) = \left(\frac{\rho}{1-\rho^2} + \frac{2\rho^3}{(1-\rho^2)^2} - \frac{2\rho}{(1-\rho^2)^2}\right) E\left(y\frac{\partial y}{\partial \beta}\right) = \frac{-\rho}{1-\rho^2} \times E\left(y\frac{\partial y}{\partial \beta}\right) \tag{165}$$

From results (101) and (256), we have

$$\begin{aligned} E\left(\frac{\partial^2 \log(f(x,w))}{\partial\gamma\partial\beta}\right) &= \frac{\rho^2}{1-\rho^2}E\left(y\frac{\partial^2 y}{\partial\gamma\partial\beta}\right) - \frac{\rho^2}{1-\rho^2}E\left(\frac{\partial y}{\partial\gamma}\frac{\partial y}{\partial\beta} + y\frac{\partial^2 y}{\partial\gamma\partial\beta}\right) \\ &\quad + 1/\gamma - E\left(w^\beta \log(w)\beta\gamma^{\beta-1}\right) - E\left(w^\beta\gamma^{\beta-1}\right) - E\left(w^\beta\beta\gamma^{\beta-1}\log(\gamma)\right) \\ &\equiv -\frac{\rho^2}{1-\rho^2}E\left(\frac{\partial y}{\partial\gamma}\frac{\partial y}{\partial\beta}\right) + 1/\gamma - T_1 - T_2 - T_3 \end{aligned} \tag{166}$$

We have

$$\begin{aligned} T_1 &= -\log(w)w^\beta\beta\gamma^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \gamma^{\beta-1}\beta\left(\beta w^{\beta-1}\log(w) + w^{\beta-1}\right)\exp\left(-(\gamma w)^\beta\right)\,dw \\ &= \frac{\beta}{\gamma}\int_0^\infty \log(w)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw + \frac{1}{\gamma}\int_0^\infty \gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\ &= \frac{\beta}{\gamma}\times E\left(\log(w)\right) + \frac{1}{\gamma} \end{aligned} \tag{167}$$

and

$$\begin{aligned} T_2 &= -\gamma^{\beta-1}w^\beta\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \gamma^{\beta-1}\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\ &= \frac{1}{\gamma} \end{aligned} \tag{168}$$

and

$$\begin{aligned} T_3 &= -\log(\gamma)\gamma^{\beta-1}\beta w^\beta\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \frac{\beta\log(\gamma)}{\gamma}\int_0^\infty \gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\ &= \frac{\beta\log(\gamma)}{\gamma} \end{aligned} \tag{169}$$

From results (166)–(169), we have

$$\begin{aligned} E\left(\frac{\partial^2 \log(f(x,w))}{\partial\gamma\partial\beta}\right) &= -\frac{\rho^2}{1-\rho^2}E\left(\frac{\partial y}{\partial\gamma}\frac{\partial y}{\partial\beta}\right) - \frac{\beta}{\gamma}\times E\left(\log(w)\right) \\ &\quad - \frac{1}{\gamma} - \frac{\beta\log(\gamma)}{\gamma} \end{aligned} \tag{170}$$

14 Appendix E3—$E\left(\frac{\partial f/\partial\theta_i}{f} \times \frac{\partial f/\partial\theta_j}{f}\right)$'s

(The existence and finiteness of the expectations involving partial derivatives of y are established in Appendix H. The existence and finiteness of $E(\log(w))$ and $E\left((\log(w))^2\right)$ follow from Lemma 2.)

Recall from (11) that the pdf of the Gaussian-Weibull is given by

$$\begin{aligned} f(x,w) \equiv\ & \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \\ & \times \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sigma\sqrt{1-\rho^2}} \times \exp\left(-\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 / \left(2\left(1-\rho^2\right)\right)\right) \end{aligned} \tag{171}$$

where

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma \times w)^\beta\right)\right)$$

Thus

$$\begin{aligned} \frac{\frac{\partial f}{\partial\mu}}{f} &= \frac{1}{\sigma} \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2} \\ \frac{\frac{\partial f}{\partial\sigma}}{f} &= \frac{-1}{\sigma} + \frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right) \\ \frac{\frac{\partial f}{\partial\rho}}{f} &= \frac{\rho}{1-\rho^2} + \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right) y}{1-\rho^2} - \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \rho}{(1-\rho^2)^2} \\ \frac{\frac{\partial f}{\partial\gamma}}{f} &= \frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1} + \left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial\gamma} \\ \frac{\frac{\partial f}{\partial\beta}}{f} &= \log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w) + \left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial\beta} \end{aligned} \tag{172}$$

From (172) and (256), we have

$$\begin{aligned} E\left(\left(\frac{\frac{\partial f}{\partial\mu}}{f}\right)^2\right) &= E\left(\left(\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{1}{\sigma}}{1-\rho^2}\right)^2\right) \\ &= \frac{1}{\sigma^2(1-\rho^2)^2} E\left(\left(\frac{x-\mu}{\sigma}\right)^2 - 2\rho y\left(\frac{x-\mu}{\sigma}\right) + \rho^2 y^2\right) \\ &= \frac{1}{\sigma^2(1-\rho^2)^2}\left(1 - 2\rho^2 + \rho^2\right) = \frac{1}{\sigma^2(1-\rho^2)} \end{aligned} \tag{173}$$

From (172), (256), (289), and (290), we have

$$
\begin{aligned}
E\left(\left(\frac{\frac{\partial f}{\partial \sigma}}{f}\right)^2\right) &= E\left(\frac{1}{\sigma^2} - \frac{2}{\sigma^2(1-\rho^2)}\left(\frac{x-\mu}{\sigma} - \rho y\right)\left(\frac{x-\mu}{\sigma}\right)\right. \\
&\quad \left. + \frac{1}{\sigma^2(1-\rho^2)^2}\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\left(\frac{x-\mu}{\sigma}\right)^2\right) \\
&= \frac{1}{\sigma^2} - \frac{2}{\sigma^2(1-\rho^2)}(1-\rho^2) \\
&\quad + \frac{1}{\sigma^2(1-\rho^2)^2} E\left(\left(\frac{x-\mu}{\sigma}\right)^4 - 2\rho y\left(\frac{x-\mu}{\sigma}\right)^3 + \rho^2 y^2\left(\frac{x-\mu}{\sigma}\right)^2\right) \\
&= \frac{-1}{\sigma^2} + \frac{1}{\sigma^2(1-\rho^2)^2}\left(3 - 6\rho^2 + \rho^2 + 2\rho^4\right) \\
&= \frac{-1}{\sigma^2} + \frac{1}{\sigma^2(1-\rho^2)^2}\left(3 - 3\rho^2 - 2\rho^2 + 2\rho^4\right) \\
&= \frac{-1}{\sigma^2} + \frac{1}{\sigma^2(1-\rho^2)}\left(1 + 2 - 2\rho^2\right) \\
&= \frac{1}{\sigma^2}\left(1 + \frac{1}{1-\rho^2}\right)
\end{aligned} \tag{174}
$$

From (172), (256), and (273)–(276), we have

$$
\begin{aligned}
E\left(\left(\frac{\frac{\partial f}{\partial \rho}}{f}\right)^2\right) &= \frac{\rho^2}{(1-\rho^2)^2} + \frac{2\rho}{(1-\rho^2)^2} E\left(\left(\frac{x-\mu}{\sigma}\right) y - \rho y^2\right) \\
&\quad - \frac{2\rho^2}{(1-\rho^2)^3} E\left(\left(\frac{x-\mu}{\sigma}\right)^2 - 2\rho y\left(\frac{x-\mu}{\sigma}\right) + \rho^2 y^2\right) \\
&\quad + \frac{1}{(1-\rho^2)^2} E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 y^2\right) - \frac{2\rho}{(1-\rho^2)^3} E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)^3 y\right) \\
&\quad + \frac{\rho^2}{(1-\rho^2)^4} E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)^4\right) \\
&= \frac{\rho^2}{(1-\rho^2)^2} + \frac{2\rho}{(1-\rho^2)^2}(\rho - \rho) - \frac{2\rho^2}{(1-\rho^2)^3}\left(1 - 2\rho^2 + \rho^2\right) \\
&\quad + \frac{1}{(1-\rho^2)^2}\left(1-\rho^2\right) - \frac{2\rho}{(1-\rho^2)^3} \times 0 + \frac{\rho^2}{(1-\rho^2)^4} 3(1-\rho^2)^2 \\
&= \frac{1+\rho^2}{(1-\rho^2)^2}
\end{aligned} \tag{175}
$$

From (172), we have

$$
\begin{aligned}
E\left(\left(\frac{\frac{\partial f}{\partial \gamma}}{f}\right)^2\right) &= E\left(\left(\frac{\beta}{\gamma} - w^\beta\beta\gamma^{\beta-1}\right)^2\right) + 2E\left(\left(\frac{\beta}{\gamma} - w^\beta\beta\gamma^{\beta-1}\right)\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \gamma}\right)\right) \\
&\quad + E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \frac{\rho^2}{(1-\rho^2)^2}\left(\frac{\partial y}{\partial \gamma}\right)^2\right) \\
&\equiv T_1 + T_2 + T_3 \qquad (176)
\end{aligned}
$$

$$
\begin{aligned}
T_1 &= \int_0^\infty \left(\frac{\beta^2}{\gamma^2} - 2w^\beta\beta^2\gamma^{\beta-2} + w^{2\beta}\beta^2\gamma^{2\beta-2}\right)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= \frac{\beta^2}{\gamma^2} - 2\int_0^\infty w^\beta\beta^2\gamma^{\beta-2}\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\quad + \int_0^\infty w^{2\beta}\beta^2\gamma^{2\beta-2}\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\equiv \frac{\beta^2}{\gamma^2} - 2U_1 + U_2 \qquad (177)
\end{aligned}
$$

$$
\begin{aligned}
U_1 &= -w^\beta\beta^2\gamma^{\beta-2}\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \gamma^{\beta-2}\beta^3 w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= \frac{\beta^2}{\gamma^2}\int_0^\infty \gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw = \frac{\beta^2}{\gamma^2} \qquad (178)
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= -w^{2\beta}\beta^2\gamma^{2\beta-2}\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty 2\beta w^{2\beta-1}\beta^2\gamma^{2\beta-2}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= \int_0^\infty 2\beta^2 w^\beta\gamma^{\beta-2}\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= -2\beta^2 w^\beta\gamma^{\beta-2}\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty 2\beta^3 w^{\beta-1}\gamma^{\beta-2}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= \frac{2\beta^2}{\gamma^2} \qquad (179)
\end{aligned}
$$

From results (177)–(179), we have

$$T_1 = \frac{\beta^2}{\gamma^2} \qquad (180)$$

From result (258), we have

$$T_2 = \int_0^\infty 2\left(\frac{\beta}{\gamma} - w^\beta\beta\gamma^{\beta-1}\right)\text{denweib}\,\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \gamma}\int_{-\infty}^\infty\left(\frac{x-\mu}{\sigma} - \rho y\right)\text{denin}\,dx\,dw = 0 \qquad (181)$$

where denin is defined in (254) and denweib is defined in (259).

From result (264), we have

$$T_3 = \int_0^\infty \text{denweib}\,\frac{\rho^2}{(1-\rho^2)^2}\left(\frac{\partial y}{\partial \gamma}^2\right)\int_{-\infty}^\infty\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\text{denin}\,dx\,dw = \frac{\rho^2}{1-\rho^2}E\left(\left(\frac{\partial y}{\partial \gamma}\right)^2\right) \qquad (182)$$

From results (176), and (180)–(182), we have

$$E\left(\left(\frac{\frac{\partial f}{\partial \gamma}}{f}\right)^2\right) = \frac{\rho^2}{1-\rho^2}\,E\left(\left(\frac{\partial y}{\partial \gamma}\right)^2\right) + \frac{\beta^2}{\gamma^2} \tag{183}$$

From (172), we have

$$\begin{aligned}
E\left(\left(\frac{\frac{\partial f}{\partial \beta}}{f}\right)^2\right) &= E\left(\left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right)^2\right) \\
&\quad + 2E\left(\left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right)\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \beta}\right)\right) \\
&\quad + E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \frac{\rho^2}{(1-\rho^2)^2}\left(\frac{\partial y}{\partial \beta}\right)^2\right) \\
&\equiv T_1 + T_2 + T_3
\end{aligned} \tag{184}$$

From result (264), we have

$$T_3 = \int_0^\infty \text{denweib}\,\frac{\rho^2}{(1-\rho^2)^2}\left(\frac{\partial y}{\partial \beta}^2\right)\int_{-\infty}^\infty \left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \text{denin}\,dx\,dw = \frac{\rho^2}{1-\rho^2} E\left(\left(\frac{\partial y}{\partial \beta}\right)^2\right) \tag{185}$$

From result (258), we have

$$T_2 = \int_0^\infty 2\left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right)\text{denweib}\,\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \beta}\int_{-\infty}^\infty \left(\frac{x-\mu}{\sigma} - \rho y\right)\text{denin}\,dx\,dw = 0 \tag{186}$$

Next, we have

$$\begin{aligned}
T_1 &= \left(\log\gamma + \frac{1}{\beta}\right) E\left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right) \\
&\quad + \left(\log\gamma + \frac{1}{\beta}\right) E\left(\log(w) - (\gamma w)^\beta \log(\gamma w)\right) \\
&\quad + E\left(\left(\log(w) - (\gamma w)^\beta \log(\gamma w)\right)^2\right) \\
&\equiv U_1 + U_2 + U_3
\end{aligned} \tag{187}$$

By result (281)

$$U_1 = 0 \tag{188}$$

and

$$U_2 = -\left(\log(\gamma) + \frac{1}{\beta}\right)^2 \tag{189}$$

Now,

$$\begin{aligned}
U_3 &= E\left((\log(w))^2\right) + E\left((\gamma w)^{2\beta}(\log(\gamma))^2\right) + E\left((\gamma w)^{2\beta}(\log(w))^2\right) \\
&\quad - 2E\left((\gamma w)^\beta \log(\gamma)\log(w)\right) - 2E\left((\gamma w)^\beta (\log(w))^2\right) + 2E\left((\gamma w)^{2\beta}\log(\gamma)\log(w)\right) \\
&\equiv V_1 + V_2 + V_3 + V_4 + V_5 + V_6
\end{aligned} \tag{190}$$

$$
\begin{aligned}
V_2 &= (\log(\gamma))^2 \left(-(\gamma w)^{2\beta} \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \gamma^{2\beta} 2\beta w^{2\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \right) \\
&= (\log(\gamma))^2 \int_0^\infty \gamma^\beta 2 w^\beta \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&= (\log(\gamma))^2 \left(-2\gamma^\beta w^\beta \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + 2\int_0^\infty \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \right) \\
&= 2\left(\log(\gamma)\right)^2 \qquad (191)
\end{aligned}
$$

$$
\begin{aligned}
V_3 &= -(\gamma w)^{2\beta} \left(\log(w)\right)^2 \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \gamma^{2\beta} 2\beta w^{2\beta-1} (\log(w))^2 \exp\left(-(\gamma w)^\beta\right)\, dw \\
&\quad + \int_0^\infty (\gamma w)^{2\beta}\, 2\log(w) \frac{1}{w} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&= \int_0^\infty \gamma^\beta 2 w^\beta (\log(w))^2 \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&\quad + \frac{1}{\beta} \int_0^\infty \gamma^\beta 2 w^\beta \log(w) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&= -2\gamma^\beta w^\beta (\log(w))^2 \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty 2\gamma^\beta \left(\beta w^{\beta-1} (\log(w))^2 + w^\beta 2\log(w) \frac{1}{w} \right) \exp\left(-(\gamma w)^\beta\right)\, dw \\
&\quad - \frac{1}{\beta} \gamma^\beta 2 w^\beta \log(w) \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \frac{2}{\beta} \gamma^\beta \left(\beta w^{\beta-1} \log(w) + w^\beta \frac{1}{w} \right) \exp\left(-(\gamma w)^\beta\right)\, dw \\
&= 2\int_0^\infty (\log(w))^2 \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&\quad + \frac{4}{\beta} \int_0^\infty \log(w) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&\quad + \frac{2}{\beta} \int_0^\infty \log(w) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&\quad + \frac{2}{\beta^2} \int_0^\infty \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&= 2E\left((\log(w))^2\right) + \frac{6}{\beta} E\left(\log(w)\right) + \frac{2}{\beta^2} \qquad (192)
\end{aligned}
$$

$$
\begin{aligned}
V_4 &= -2\log(\gamma) \left(-(\gamma w)^\beta \log(w) \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \log(w) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \right. \\
&\quad \left. + \frac{1}{\beta} \int_0^\infty \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \right) \\
&= -2\log(\gamma) \left(E\left(\log(w)\right) + \frac{1}{\beta} \right) \qquad (193)
\end{aligned}
$$

$$
\begin{aligned}
V_5 &= -2\left(-(\gamma w)^\beta (\log(w))^2 \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty \right.\\
&\quad \left. + \int_0^\infty \gamma^\beta \left(\beta w^{\beta-1}(\log(w))^2 + w^\beta 2\log(w)\frac{1}{w}\right)\exp\left(-(\gamma w)^\beta\right)\, dw\right)\\
&= -2\left(\int_0^\infty (\log(w))^2 \gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\, dw + \frac{2}{\beta}\int_0^\infty \log(w)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\, dw\right)\\
&= -2E\left((\log(w))^2\right) - \frac{4}{\beta}E\left(\log(w)\right) \qquad (194)
\end{aligned}
$$

$$
\begin{aligned}
V_6 &= 2\log(\gamma)\left(-(\gamma w)^{2\beta}\log(w)\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty\right.\\
&\quad \left. + \int_0^\infty \gamma^{2\beta}\left(2\beta w^{2\beta-1}\log(w) + w^{2\beta-1}\right)\exp\left(-(\gamma w)^\beta\right)\, dw\right)\\
&= 2\log(\gamma)\left(\int_0^\infty 2\gamma^\beta w^\beta \log(w)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\, dw\right.\\
&\quad \left. + \int_0^\infty \gamma^\beta \frac{1}{\beta} w^\beta \gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\, dw\right)\\
&= 2\log(\gamma)\left(-2\gamma^\beta w^\beta \log(w)\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty\right.\\
&\quad + 2\int_0^\infty \gamma^\beta\left(\beta w^{\beta-1}\log(w) + w^{\beta-1}\right)\exp\left(-(\gamma w)^\beta\right)\, dw\\
&\quad - \gamma^\beta\frac{1}{\beta}w^\beta\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty\\
&\quad \left. + \frac{1}{\beta}\int_0^\infty \gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\, dw\right)\\
&= 2\log(\gamma)\left(2E\left(\log(w)\right) + \frac{2}{\beta} + \frac{1}{\beta}\right)\\
&= 4\log(\gamma)E\left(\log(w)\right) + \frac{6\log(\gamma)}{\beta} \qquad (195)
\end{aligned}
$$

From results (190)–(195), we have

$$
\begin{aligned}
U_3 &= E\left((\log(w))^2\right) + 2\left(\log(\gamma)\right)^2\\
&\quad + 2E\left((\log(w))^2\right) + \frac{6}{\beta}E\left(\log(w)\right) + \frac{2}{\beta^2}\\
&\quad - 2\log(\gamma)\left(E\left(\log(w)\right) + \frac{1}{\beta}\right) - 2E\left((\log(w))^2\right) - \frac{4}{\beta}E\left(\log(w)\right)\\
&\quad + 4\log(\gamma)E\left(\log(w)\right) + \frac{6\log(\gamma)}{\beta}\\
&= E\left((\log(w))^2\right) + \frac{2}{\beta}E\left(\log(w)\right) + \frac{4\log(\gamma)}{\beta}\\
&\quad + 2\log(\gamma)\left(E\left(\log(w)\right)\right) + 2\left(\log(\gamma)\right)^2 + \frac{2}{\beta^2} \qquad (196)
\end{aligned}
$$

Results (187)–(189) and (196) yield

$$T_1 = E\left((\log(w))^2\right) + \frac{2}{\beta} E(\log(w)) + \frac{2\log(\gamma)}{\beta} + 2\log(\gamma)\left(E(\log(w))\right) + (\log(\gamma))^2 + \frac{1}{\beta^2} \tag{197}$$

Results (184)–(186) and (197) yield

$$\begin{aligned} E\left(\left(\frac{\frac{\partial f}{\partial \beta}}{f}\right)^2\right) &= \frac{\rho^2}{1-\rho^2} E\left(\left(\frac{\partial y}{\partial \beta}\right)^2\right) \\ &\quad + E\left((\log(w))^2\right) + \frac{2}{\beta} E(\log(w)) + \frac{2\log(\gamma)}{\beta} \\ &\quad + 2\log(\gamma)\left(E(\log(w))\right) + (\log(\gamma))^2 + \frac{1}{\beta^2} \end{aligned} \tag{198}$$

From results (172), (258), and (265), we have

$$\begin{aligned} E\left(\frac{\frac{\partial f}{\partial \mu}}{f} \times \frac{\frac{\partial f}{\partial \sigma}}{f}\right) &= E\left(\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2} \times \left(\frac{-1}{\sigma} + \frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\right)\right) \\ &= 0 \end{aligned} \tag{199}$$

From results (172), (258), (266), and (267), we have

$$\begin{aligned} E\left(\frac{\frac{\partial f}{\partial \mu}}{f} \times \frac{\frac{\partial f}{\partial \rho}}{f}\right) &= E\left(\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2} \times \left(\frac{\rho}{1-\rho^2} + \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right) y}{1-\rho^2} - \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \rho}{(1-\rho^2)^2}\right)\right) \\ &= 0 \end{aligned} \tag{200}$$

From results (172), (258), and (264), we have

$$\begin{aligned} E\left(\frac{\frac{\partial f}{\partial \mu}}{f} \times \frac{\frac{\partial f}{\partial \gamma}}{f}\right) &= E\left(\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2} \times \left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1} + \left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \gamma}\right)\right) \\ &= \frac{\rho}{\sigma(1-\rho^2)} E\left(\frac{\partial y}{\partial \gamma}\right) \end{aligned} \tag{201}$$

From results (172), (258), and (264), we have

$$\begin{aligned} E\left(\frac{\frac{\partial f}{\partial \mu}}{f} \times \frac{\frac{\partial f}{\partial \beta}}{f}\right) &= E\left(\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\right. \\ &\quad \left. \times \left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w) + \left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \beta}\right)\right) \\ &= \frac{\rho}{\sigma(1-\rho^2)} E\left(\frac{\partial y}{\partial \beta}\right) \end{aligned} \tag{202}$$

Next, from result (172), we have

$$
\begin{aligned}
E\left(\frac{\frac{\partial f}{\partial \sigma}}{f} \times \frac{\frac{\partial f}{\partial \rho}}{f}\right) &= E\left(\left(\frac{-1}{\sigma} + \frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\right)\right. \\
&\quad \left.\times\left(\frac{\rho}{1-\rho^2} + \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)y}{1-\rho^2} - \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\rho}{(1-\rho^2)^2}\right)\right) \\
&= E\left(\frac{-1}{\sigma}\left(\frac{\rho}{1-\rho^2} + \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)y}{1-\rho^2} - \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\rho}{(1-\rho^2)^2}\right)\right) \\
&\quad + E\left(\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\frac{\rho}{1-\rho^2}\right) \\
&\quad + E\left(\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)y}{1-\rho^2}\right) \\
&\quad - E\left(\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\rho}{(1-\rho^2)^2}\right) \\
&\equiv T_1 + T_2 + T_3 + T_4
\end{aligned} \tag{203}
$$

Results (258) and (264) yield

$$
T_1 = 0 \tag{204}
$$

The fact that $(x-\mu)/\sigma$ and y are bivariate standard normals with correlation ρ yields

$$
T_2 = \frac{\rho}{\sigma(1-\rho^2)} \tag{205}
$$

By results (289) and (290), we have

$$
\begin{aligned}
T_3 &= \frac{1}{\sigma(1-\rho^2)^2}\left(E\left(\left(\frac{x-\mu}{\sigma}\right)^3 y\right) - 2\rho E\left(\left(\frac{x-\mu}{\sigma}\right)^2 y^2\right) + \rho^2 E\left(\left(\frac{x-\mu}{\sigma}\right)y^3\right)\right) \\
&= \frac{1}{\sigma(1-\rho^2)^2}\left(3\rho - 2\rho(1+2\rho^2) + 3\rho^3\right) \\
&= \frac{1}{\sigma(1-\rho^2)^2} \times \rho(1-\rho^2) = \frac{\rho}{\sigma(1-\rho^2)}
\end{aligned} \tag{206}
$$

By result (268), we have

$$
T_4 = \frac{-\rho}{\sigma(1-\rho^2)^3} E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)^3\left(\frac{x-\mu}{\sigma}\right)\right) = \frac{-3\rho}{\sigma(1-\rho^2)} \tag{207}
$$

Results (203)–(207) yield

$$
E\left(\frac{\frac{\partial f}{\partial \sigma}}{f} \times \frac{\frac{\partial f}{\partial \rho}}{f}\right) = \frac{-\rho}{\sigma(1-\rho^2)} \tag{208}
$$

Next, from result (172), we have

$$\begin{aligned}
E\left(\frac{\frac{\partial f}{\partial\sigma}}{f}\times\frac{\frac{\partial f}{\partial\gamma}}{f}\right) &= E\left(\left(\frac{-1}{\sigma}+\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma}-\rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\right)\right.\\
&\quad\left.\times\left(\frac{\beta}{\gamma}-w^\beta\beta\gamma^{\beta-1}+\left(\frac{x-\mu}{\sigma}-\rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial\gamma}\right)\right)\\
&= E\left(\frac{-1}{\sigma}\left(\frac{\beta}{\gamma}-w^\beta\beta\gamma^{\beta-1}\right)\right)\\
&\quad-\frac{\rho}{\sigma(1-\rho^2)}E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)\frac{\partial y}{\partial\gamma}\right)\\
&\quad+E\left(\frac{1}{\sigma(1-\rho^2)}\left(\left(\frac{x-\mu}{\sigma}\right)^2-\rho\left(\frac{x-\mu}{\sigma}\right)y\right)\left(\frac{\beta}{\gamma}-w^\beta\beta\gamma^{\beta-1}\right)\right)\\
&\quad+\frac{\rho}{\sigma(1-\rho^2)^2}E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)^2\left(\frac{x-\mu}{\sigma}\right)\frac{\partial y}{\partial\gamma}\right)\\
&\equiv T_1+T_2+T_3+T_4 \qquad (209)
\end{aligned}$$

Result (277) yields

$$T_1=0 \qquad (210)$$

Result (258) yields

$$T_2=0 \qquad (211)$$

Results (256), (263), and (277) yield

$$T_3 \;=\; 0 \qquad (212)$$

Result (270) yields

$$T_4 \;=\; \frac{\rho^2}{\sigma(1-\rho^2)}E\left(y\frac{\partial y}{\partial\gamma}\right) \qquad (213)$$

Results (209)–(213) yield

$$E\left(\frac{\frac{\partial f}{\partial\sigma}}{f}\times\frac{\frac{\partial f}{\partial\gamma}}{f}\right) \;=\; \frac{\rho^2}{\sigma(1-\rho^2)}E\left(y\frac{\partial y}{\partial\gamma}\right) \qquad (214)$$

Next, from result (172), we have

$$
\begin{aligned}
E\left(\frac{\frac{\partial f}{\partial\sigma}}{f}\times\frac{\frac{\partial f}{\partial\beta}}{f}\right) &= E\left(\left(\frac{-1}{\sigma}+\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma}-\rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\right)\right. \\
&\quad \left.\times\left(\log\gamma+\frac{1}{\beta}+\log(w)-(\gamma w)^\beta\log(\gamma w)+\left(\frac{x-\mu}{\sigma}-\rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial\beta}\right)\right) \\
&= E\left(\frac{-1}{\sigma}\left(\log\gamma+\frac{1}{\beta}+\log(w)-(\gamma w)^\beta\log(\gamma w)\right)\right) \\
&\quad -\frac{\rho}{\sigma(1-\rho^2)}E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)\frac{\partial y}{\partial\beta}\right) \\
&\quad +\frac{1}{\sigma(1-\rho^2)}E\left(\left(\log\gamma+\frac{1}{\beta}+\log(w)-(\gamma w)^\beta\log(\gamma w)\right)\left(\frac{x-\mu}{\sigma}-\rho y\right)\left(\frac{x-\mu}{\sigma}\right)\right) \\
&\quad +\frac{\rho}{\sigma(1-\rho^2)^2}E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)^2\left(\frac{x-\mu}{\sigma}\right)\frac{\partial y}{\partial\beta}\right) \\
&\equiv T_1+T_2+T_3+T_4
\end{aligned} \tag{215}
$$

Result (281) yields

$$T_1 = 0 \tag{216}$$

Result (258) yields

$$T_2 = 0 \tag{217}$$

Results (256), (263), and (281) yield

$$T_3 = 0 \tag{218}$$

Result (270) yields

$$T_4 = \frac{\rho^2}{\sigma(1-\rho^2)}E\left(y\frac{\partial y}{\partial\beta}\right) \tag{219}$$

Results (215)–(219) yield

$$E\left(\frac{\frac{\partial f}{\partial\sigma}}{f}\times\frac{\frac{\partial f}{\partial\beta}}{f}\right) = \frac{\rho^2}{\sigma(1-\rho^2)}E\left(y\frac{\partial y}{\partial\beta}\right) \tag{220}$$

Next, from result (172), we have

$$\begin{aligned}
E\left(\frac{\frac{\partial f}{\partial \rho}}{f} \times \frac{\frac{\partial f}{\partial \gamma}}{f}\right) &= E\left(\left(\frac{\rho}{1-\rho^2} + \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right) y}{1-\rho^2} - \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \rho}{(1-\rho^2)^2}\right)\right. \\
&\quad \left. \times \left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1} + \left(\frac{x-\mu}{\sigma} - \rho y\right) \frac{\rho}{1-\rho^2} \frac{\partial y}{\partial \gamma}\right)\right) \\
&= \frac{\rho}{1-\rho^2} E\left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1} + \left(\frac{x-\mu}{\sigma} - \rho y\right) \frac{\rho}{1-\rho^2} \frac{\partial y}{\partial \gamma}\right) \\
&\quad + E\left(\left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1}\right)\left(\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right) y}{1-\rho^2}\right)\right) \\
&\quad + E\left(\frac{\rho y}{(1-\rho^2)^2} \frac{\partial y}{\partial \gamma}\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\right) \\
&\quad - \frac{\rho}{(1-\rho^2)^2} E\left(\left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1}\right)\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\right) \\
&\quad - \frac{\rho^2}{(1-\rho^2)^3} E\left(\frac{\partial y}{\partial \gamma}\left(\frac{x-\mu}{\sigma} - \rho y\right)^3\right) \\
&\equiv T_1 + T_2 + T_3 + T_4 + T_5
\end{aligned} \tag{221}$$

Results (258) and (277) yield

$$T_1 = 0 \tag{222}$$

Result (258) yields

$$T_2 = 0 \tag{223}$$

Result (264) yields

$$T_3 = \frac{\rho}{1-\rho^2} E\left(y \frac{\partial y}{\partial \gamma}\right) \tag{224}$$

Results (264) and (277) yield

$$T_4 = 0 \tag{225}$$

Result (274) yields

$$T_5 = 0 \tag{226}$$

Results (221)–(226) yield

$$E\left(\frac{\frac{\partial f}{\partial \rho}}{f} \times \frac{\frac{\partial f}{\partial \gamma}}{f}\right) = \frac{\rho}{1-\rho^2} E\left(y \frac{\partial y}{\partial \gamma}\right) \tag{227}$$

Next, from result (172), we have

$$
\begin{aligned}
E\left(\frac{\frac{\partial f}{\partial \rho}}{f} \times \frac{\frac{\partial f}{\partial \beta}}{f}\right) &= E\Bigg(\left(\frac{\rho}{1-\rho^2} + \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right) y}{1-\rho^2} - \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \rho}{(1-\rho^2)^2}\right) \\
&\quad \times \left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w) + \left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \beta}\right)\Bigg) \\
&= \frac{\rho}{1-\rho^2} E\left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w) + \left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \beta}\right) \\
&\quad + E\left(\left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right)\left(\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right) y}{1-\rho^2}\right)\right) \\
&\quad + E\left(\frac{\rho y}{(1-\rho^2)^2}\frac{\partial y}{\partial \beta}\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\right) \\
&\quad - \frac{\rho}{(1-\rho^2)^2} E\left(\left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right)\left(\frac{x-\mu}{\sigma} - \rho y\right)^2\right) \\
&\quad - \frac{\rho^2}{(1-\rho^2)^3} E\left(\frac{\partial y}{\partial \beta}\left(\frac{x-\mu}{\sigma} - \rho y\right)^3\right) \\
&\equiv T_1 + T_2 + T_3 + T_4 + T_5
\end{aligned}
\tag{228}
$$

Results (258) and (281) yield

$$T_1 = 0 \tag{229}$$

Result (258) yields

$$T_2 = 0 \tag{230}$$

Result (264) yields

$$T_3 = \frac{\rho}{1-\rho^2} E\left(y\frac{\partial y}{\partial \beta}\right) \tag{231}$$

Results (264) and (281) yield

$$T_4 = 0 \tag{232}$$

Result (274) yields

$$T_5 = 0 \tag{233}$$

Results (228)–(233) yield

$$E\left(\frac{\frac{\partial f}{\partial \rho}}{f} \times \frac{\frac{\partial f}{\partial \beta}}{f}\right) = \frac{\rho}{1-\rho^2} E\left(y\frac{\partial y}{\partial \beta}\right) \tag{234}$$

Next, from result (172), we have

$$
\begin{aligned}
E\left(\frac{\frac{\partial f}{\partial \gamma}}{f} \times \frac{\frac{\partial f}{\partial \beta}}{f}\right) &= E\left(\left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1} + \left(\frac{x-\mu}{\sigma} - \rho y\right) \frac{\rho}{1-\rho^2} \frac{\partial y}{\partial \gamma}\right)\right. \\
&\quad \left. \times \left(\log \gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w) + \left(\frac{x-\mu}{\sigma} - \rho y\right) \frac{\rho}{1-\rho^2} \frac{\partial y}{\partial \beta}\right)\right) \\
&= E\left(\left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1}\right) \left(\frac{x-\mu}{\sigma} - \rho y\right) \frac{\rho}{1-\rho^2} \frac{\partial y}{\partial \beta}\right) \\
&\quad + E\left(\left(\log \gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right) \left(\frac{x-\mu}{\sigma} - \rho y\right) \frac{\rho}{1-\rho^2} \frac{\partial y}{\partial \gamma}\right) \\
&\quad + \frac{\rho^2}{(1-\rho^2)^2} E\left(\frac{\partial y}{\partial \gamma} \frac{\partial y}{\partial \beta} \left(\frac{x-\mu}{\sigma} - \rho y\right)^2\right) \\
&\quad + E\left(\left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1}\right) \left(\log \gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right)\right) \\
&\equiv T_1 + T_2 + T_3 + T_4 \qquad (235)
\end{aligned}
$$

Result (258) yields

$$T_1 = 0 \qquad (236)$$

and

$$T_2 = 0 \qquad (237)$$

Result (264) yields

$$T_3 = \frac{\rho^2}{1-\rho^2} E\left(\frac{\partial y}{\partial \gamma} \frac{\partial y}{\partial \beta}\right) \qquad (238)$$

Now note that

$$
\begin{aligned}
T_4 &= \int_0^\infty \left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1}\right) \left(\log \gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) dw \\
&= \frac{\beta}{\gamma}\left(\log \gamma + \frac{1}{\beta}\right) \\
&\quad - \left(\log \gamma + \frac{1}{\beta}\right) \int_0^\infty w^\beta \beta \gamma^{\beta-1} \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) dw \\
&\quad + \frac{\beta}{\gamma} \int_0^\infty \left(\log(w) - (\gamma w)^\beta \log(\gamma w)\right) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) dw \\
&\quad - \int_0^\infty w^\beta \beta \gamma^{\beta-1} \left(\log(w) - (\gamma w)^\beta \log(\gamma w)\right) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) dw \\
&\equiv U_1 + U_2 + U_3 + U_4 \qquad (239)
\end{aligned}
$$

We have

$$
\begin{aligned}
\int_0^\infty w^\beta \beta \gamma^{\beta-1} \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) dw &= -w^\beta \beta \gamma^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty \\
&\quad + \frac{\beta}{\gamma} \int_0^\infty \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) dw \\
&= \frac{\beta}{\gamma}
\end{aligned}
$$

Thus

$$U_2 = -\left(\log\gamma + \frac{1}{\beta}\right)\frac{\beta}{\gamma} \tag{240}$$

By results (279) and (280), we have

$$U_3 = -\frac{\beta}{\gamma}\left(\log\gamma + \frac{1}{\beta}\right) \tag{241}$$

Next

$$\begin{aligned}
U_4 &= -\int_0^\infty w^\beta\beta\gamma^{\beta-1}\log(w)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\quad + \int_0^\infty w^\beta\beta\gamma^{\beta-1}(\gamma w)^\beta\log(\gamma w)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\equiv V_1 + V_2
\end{aligned} \tag{242}$$

We have

$$\begin{aligned}
-V_1 &= -w^\beta\beta\gamma^{\beta-1}\log(w)\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty \\
&\quad + \int_0^\infty \beta\gamma^{\beta-1}\left(\beta w^{\beta-1}\log(w) + w^{\beta-1}\right)\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= \frac{\beta}{\gamma}\int_0^\infty \log(w)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\quad + \frac{1}{\gamma}\int_0^\infty \gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= \frac{\beta}{\gamma}E\left(\log(w)\right) + \frac{1}{\gamma}
\end{aligned} \tag{243}$$

Next,

$$\begin{aligned}
V_2 &= \int_0^\infty \gamma^{2\beta-1}w^{2\beta}\beta\log(\gamma)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\quad + \int_0^\infty \gamma^{2\beta-1}w^{2\beta}\beta\log(w)\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\equiv W_1 + W_2
\end{aligned} \tag{244}$$

We have

$$\begin{aligned}
W_1 &= -\gamma^{2\beta-1}w^{2\beta}\beta\log(\gamma)\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + 2\beta\log(\gamma)\int_0^\infty \gamma^{\beta-1}w^\beta\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= 2\beta\log(\gamma)\left(-\gamma^{\beta-1}w^\beta\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \gamma^{\beta-1}\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw\right) \\
&= \frac{2\beta\log(\gamma)}{\gamma}
\end{aligned} \tag{245}$$

Also,

$$\begin{aligned} W_2 &= -\gamma^{2\beta-1} w^{2\beta} \beta \log(w) \exp\left(-(\gamma w)^{\beta}\right)\Big|_0^{\infty} \\ &\quad + \int_0^{\infty} \gamma^{2\beta-1} \beta \left(2\beta w^{2\beta-1}\log(w) + w^{2\beta-1}\right) \exp\left(-(\gamma w)^{\beta}\right)\, dw \\ &= 2\beta \int_0^{\infty} \gamma^{\beta-1} w^{\beta} \log(w) \gamma^{\beta} \beta w^{\beta-1} \exp\left(-(\gamma w)^{\beta}\right)\, dw \\ &\quad + \int_0^{\infty} \gamma^{\beta-1} w^{\beta} \gamma^{\beta} \beta w^{\beta-1} \exp\left(-(\gamma w)^{\beta}\right)\, dw \\ &\equiv X_1 + X_2 \end{aligned} \tag{246}$$

We have

$$\begin{aligned} X_1 &= 2\beta \left(-\gamma^{\beta-1} w^{\beta} \log(w) \exp\left(-(\gamma w)^{\beta}\right)\Big|_0^{\infty} + \int_0^{\infty} \gamma^{\beta-1} \left(\beta w^{\beta-1}\log(w) + w^{\beta-1}\right) \exp\left(-(\gamma w)^{\beta}\right)\, dw \right) \\ &= 2\beta \left(E\left(\frac{1}{\gamma}\log(w)\right) + \frac{1}{\beta\gamma} \right) = 2\left(\frac{\beta}{\gamma} E\left(\log(w)\right) + \frac{1}{\gamma}\right) \end{aligned} \tag{247}$$

and

$$X_2 = -\gamma^{\beta-1} w^{\beta} \exp\left(-(\gamma w)^{\beta}\right)\Big|_0^{\infty} + \int_0^{\infty} \gamma^{\beta-1} \beta w^{\beta-1} \exp\left(-(\gamma w)^{\beta}\right)\, dw = \frac{1}{\gamma} \tag{248}$$

Results (246)–(248) yield

$$W_2 = 2\left(\frac{\beta}{\gamma} E\left(\log(w)\right) + \frac{1}{\gamma}\right) + \frac{1}{\gamma} \tag{249}$$

Results (244), (245), and (249) yield

$$V_2 = 2\left(\frac{\beta}{\gamma} E\left(\log(w)\right) + \frac{1}{\gamma}\right) + \frac{1}{\gamma} + \frac{2\beta\log(\gamma)}{\gamma} \tag{250}$$

Results (242), (243), and (250) yield

$$U_4 = \frac{\beta}{\gamma} E\left(\log(w)\right) + \frac{1}{\gamma} + \frac{1}{\gamma} + \frac{2\beta\log(\gamma)}{\gamma} \tag{251}$$

Results (239), (240), (241), and (251) yield

$$\begin{aligned} T_4 &= -\frac{\beta}{\gamma}\left(\log(\gamma) + \frac{1}{\beta}\right) + \frac{\beta}{\gamma} E\left(\log(w)\right) + \frac{1}{\gamma} + \frac{1}{\gamma} + \frac{2\beta\log(\gamma)}{\gamma} \\ &= \frac{\beta}{\gamma} E\left(\log(w)\right) + \frac{1}{\gamma} + \frac{\beta\log(\gamma)}{\gamma} \end{aligned} \tag{252}$$

Results (235)–(238) and (252) yield

$$\begin{aligned} E\left(\frac{\frac{\partial f}{\partial\gamma}}{f} \times \frac{\frac{\partial f}{\partial\beta}}{f}\right) &= \frac{\rho^2}{1-\rho^2} E\left(\frac{\partial y}{\partial\gamma}\frac{\partial y}{\partial\beta}\right) \\ &\quad + \frac{\beta}{\gamma} E\left(\log(w)\right) + \frac{1}{\gamma} + \frac{\beta\log(\gamma)}{\gamma} \end{aligned} \tag{253}$$

15 Appendix F—Miscellaneous useful integrals

(The existence and finiteness of the expectations involving partial derivatives of y are established in Appendix H. The existence and finiteness of $E(\log(w))$ and $E\left((\log(w))^2\right)$ follow from Lemma 2.)

Define

$$\text{denin} \equiv \frac{1}{\sqrt{2\pi}} \times \frac{1}{\sigma\sqrt{1-\rho^2}} \times \exp\left(-\left(\frac{x-\mu}{\sigma}-\rho y\right)^2 / \left(2\left(1-\rho^2\right)\right)\right) \tag{254}$$

Then, because denin is simply the pdf of a $\mathrm{N}\big(\mu+\rho\sigma y, \sigma^2(1-\rho)^2\big)$,

$$\int_{-\infty}^{\infty} \text{denin}\, dx \quad = \quad 1 \tag{255}$$

(Note that the demonstrations of some of the results below could also be based on the fact that denin is the pdf of a $\mathrm{N}\big(\mu+\rho\sigma y, \sigma^2(1-\rho)^2\big)$. However, instead, in most cases we have found it expedient to explicitly perform some of the integral calculations. Either approach is valid.)

Next

$$\begin{aligned}
\int_{-\infty}^{\infty} (x-\mu)/\sigma \times \text{denin}\, dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \times \frac{(x-\mu)/\sigma}{\sigma\sqrt{1-\rho^2}} \times \exp\left(-\left(\frac{x-\mu}{\sigma}-\rho y\right)^2 / \left(2\left(1-\rho^2\right)\right)\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \times \frac{x}{\sqrt{1-\rho^2}} \times \exp\left(-\left(x-\rho y\right)^2 / \left(2\left(1-\rho^2\right)\right)\right) dx \\
&= \sqrt{1-\rho^2} \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \times \exp\left(-\left(x-\rho y/\sqrt{1-\rho^2}\right)^2 /2\right) dx \\
&= \sqrt{1-\rho^2} \times \rho y/\sqrt{1-\rho^2} = \rho y
\end{aligned} \tag{256}$$

and

$$\int_{-\infty}^{\infty} (x-\mu)/\sigma \times y \times \text{denin}\, dx = \rho y^2 \tag{257}$$

From results (255) and (256) we have

$$\int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma} - \rho y\right) \text{denin}\, dx = \rho y - \rho y = 0 \tag{258}$$

Define

$$\text{denweib} \equiv \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \tag{259}$$

Then from (11)

$$\text{gaussweib}(x, w; \mu, \sigma, \rho, \gamma, \beta) = \text{denweib} \times \text{denin} \tag{260}$$

and by result (256)

$$\begin{aligned}
E\left(\frac{x-\mu}{\sigma} \times \frac{\partial y}{\partial \gamma}\right) &= \int_0^{\infty} \text{denweib} \int_{-\infty}^{\infty} \frac{x-\mu}{\sigma} \times \frac{\partial y}{\partial \gamma} \times \text{denin}\, dx\, dw \qquad (261) \\
&= \int_0^{\infty} \text{denweib} \times \rho y \frac{\partial y}{\partial \gamma}\, dw \\
&= E\left(\rho y \frac{\partial y}{\partial \gamma}\right)
\end{aligned}$$

Similarly,

$$E\left(\frac{x-\mu}{\sigma}\times\frac{\partial y}{\partial\beta}\right)=E\left(\rho y\frac{\partial y}{\partial\beta}\right) \tag{262}$$

Next, as in the development of result (256),

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{x-\mu}{\sigma}\right)^2\times\text{denin}\,dx &= (1-\rho^2)\int_{-\infty}^{\infty}\frac{x^2}{\sqrt{2\pi}}\times\exp\left(-\left(x-\rho y/\sqrt{1-\rho^2}\right)^2/2\right)\,dx\\ &= (1-\rho^2)\times\left(1+\frac{\rho^2y^2}{1-\rho^2}\right)=1-\rho^2+\rho^2y^2 \qquad (263)\end{aligned}$$

From results (256) and (263) we have

$$\begin{aligned}\int_{-\infty}^{\infty}\left(\frac{x-\mu}{\sigma}-\rho y\right)^2\times\text{denin}\,dx &= \int_{-\infty}^{\infty}\left(\left(\frac{x-\mu}{\sigma}\right)^2-2\rho y\left(\frac{x-\mu}{\sigma}\right)+\rho^2y^2\right)\text{denin}\,dx\\ &= 1-\rho^2+\rho^2y^2-2\rho^2y^2+\rho^2y^2=1-\rho^2 \qquad (264)\end{aligned}$$

By result (288), we have

$$E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)^2\left(\frac{x-\mu}{\sigma}\right)\right)=E\left(\left(\frac{x-\mu}{\sigma}\right)^3-2\rho\left(\frac{x-\mu}{\sigma}\right)^2y+\rho^2\left(\frac{x-\mu}{\sigma}\right)y^2\right)=0 \tag{265}$$

and

$$E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)^2y\right)=E\left(\left(\frac{x-\mu}{\sigma}\right)^2y-2\rho\left(\frac{x-\mu}{\sigma}\right)y^2+\rho^2y^3\right)=0 \tag{266}$$

and

$$E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)^3\right)=E\left(\left(\frac{x-\mu}{\sigma}\right)^3-3\rho\left(\frac{x-\mu}{\sigma}\right)^2y+3\rho^2\left(\frac{x-\mu}{\sigma}\right)y^2-\rho^3E\left(y^3\right)\right)=0 \tag{267}$$

By results (289) and (290), we have

$$\begin{aligned}E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)^3\left(\frac{x-\mu}{\sigma}\right)\right) &= E\left(\left(\frac{x-\mu}{\sigma}\right)^4-3\left(\frac{x-\mu}{\sigma}\right)^3\rho y\right.\\ &\qquad\left.+3\left(\frac{x-\mu}{\sigma}\right)^2\rho^2y^2-\left(\frac{x-\mu}{\sigma}\right)\rho^3y^3\right)\\ &= 3-9\rho^2+3(1+2\rho^2)\rho^2-3\rho^4\\ &= 3-6\rho^2+3\rho^4=3-3\rho^2-3\rho^2+3\rho^4\\ &= 3(1-\rho^2)-3\rho^2(1-\rho^2)=3(1-\rho^2)^2 \qquad (268)\end{aligned}$$

Next, as in the development of (256), we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^3 \text{denin}\, dx &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \times \frac{x^3}{\sqrt{1-\rho^2}} \times \exp\left(-\left(x-\rho y\right)^2 / \left(2\left(1-\rho^2\right)\right)\right) dx \\
&= (1-\rho^2)^{3/2} \int_{-\infty}^{\infty} \frac{x^3}{\sqrt{2\pi}} \times \exp\left(-\left(x-\rho y/\sqrt{1-\rho^2}\right)^2/2\right) dx \\
&= (1-\rho^2)^{3/2} \int_{-\infty}^{\infty} \left(x^3 + \frac{3\rho y x^2}{\sqrt{1-\rho^2}} + \frac{3\rho^2 y^2 x}{1-\rho^2} + \frac{\rho^3 y^3}{(1-\rho^2)^{3/2}}\right) \phi(x)\, dx \\
&= (1-\rho^2)^{3/2} \left(\frac{3\rho y}{\sqrt{1-\rho^2}} + \frac{\rho^3 y^3}{(1-\rho^2)^{3/2}}\right) \\
&= 3\rho(1-\rho^2)y + \rho^3 y^3 \qquad (269)
\end{aligned}
$$

From results (256), (263), and (269), we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \left(\frac{x-\mu}{\sigma}\right) \times \text{denin}\, dx \\
&= \int_{-\infty}^{\infty} \left(\left(\frac{x-\mu}{\sigma}\right)^3 - 2\rho\left(\frac{x-\mu}{\sigma}\right)^2 y + \rho^2\left(\frac{x-\mu}{\sigma}\right) y^2\right) \times \text{denin}\, dx \\
&= 3\rho(1-\rho^2)y + \rho^3 y^3 - 2\rho y(1-\rho^2+\rho^2 y^2) + \rho^2 y^2(\rho y) \\
&= \rho(1-\rho^2)y \qquad (270)
\end{aligned}
$$

Next, by result (261),

$$
E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\partial y}{\partial \gamma}\right) = E\left(\frac{x-\mu}{\sigma}\frac{\partial y}{\partial \gamma}\right) - E\left(\rho y \frac{\partial y}{\partial \gamma}\right) = 0 \qquad (271)
$$

Similarly,

$$
E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\partial y}{\partial \beta}\right) = 0 \qquad (272)
$$

Next,

$$
\begin{aligned}
E\left(\left(\frac{x-\mu}{\sigma} - \rho y\right)^4\right) &= \int_0^{\infty} \text{denweib} \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma} - \rho y\right)^4 \text{denin}\, dx\, dw \\
&= \int_0^{\infty} \text{denweib} \int_{-\infty}^{\infty} \frac{(x-\rho y)^4}{\sqrt{2\pi}} \times \frac{1}{\sqrt{1-\rho^2}} \times \exp\left(-\left(x-\rho y\right)^2 / \left(2\left(1-\rho^2\right)\right)\right) dx\, dw \\
&= \int_0^{\infty} \text{denweib} \int_{-\infty}^{\infty} \frac{\left(\sqrt{1-\rho^2}\, x - \rho y\right)^4}{\sqrt{2\pi}} \times \exp\left(-\left(x-\rho y/\sqrt{1-\rho^2}\right)^2/2\right) dx\, dw \\
&= \int_0^{\infty} \text{denweib} \int_{-\infty}^{\infty} \frac{\left(\sqrt{1-\rho^2}\left(x+\rho y/\sqrt{1-\rho^2}\right) - \rho y\right)^4}{\sqrt{2\pi}} \times \exp\left(-x^2/2\right) dx\, dw \\
&= \left(1-\rho^2\right)^2 \int_0^{\infty} \text{denweib} \int_{-\infty}^{\infty} \frac{x^4}{\sqrt{2\pi}} \times \exp\left(-x^2/2\right) dx\, dw \\
&= 3\left(1-\rho^2\right)^2 \qquad (273)
\end{aligned}
$$

Similarly,

$$\int_{-\infty}^{\infty}\left(\frac{x-\mu}{\sigma}-\rho y\right)^3 \text{denin}\,dx = (1-\rho^2)^{3/2}\int_{-\infty}^{\infty}\frac{x^3}{\sqrt{2\pi}}\times\exp\left(-x^2/2\right)\,dx = 0 \tag{274}$$

and

$$E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)^3 y\right) = \left(1-\rho^2\right)^{3/2}\int_0^{\infty}\text{denweib}\int_{-\infty}^{\infty}\frac{x^3 y}{\sqrt{2\pi}}\times\exp\left(-x^2/2\right)\,dx\,dw = 0 \tag{275}$$

and (the result below can also be obtained by noting that $(x-\mu)/\sigma$ and y are distributed as standard normals with correlation ρ and then making use of results (289) and (290))

$$\begin{aligned}
E\left(\left(\frac{x-\mu}{\sigma}-\rho y\right)^2 y^2\right) &= \left(1-\rho^2\right)\int_0^{\infty}\text{denweib}\int_{-\infty}^{\infty}\frac{x^2y^2}{\sqrt{2\pi}}\times\exp\left(-x^2/2\right)\,dx\,dw \\
&= \left(1-\rho^2\right)\int_0^{\infty}\text{denweib}\times y^2\,dw \\
&= \left(1-\rho^2\right)\int_0^{\infty}\gamma^{\beta}\beta w^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\times\left(\Phi^{-1}\left(1-\exp\left(-(\gamma w)^{\beta}\right)\right)\right)^2\,dw \\
&= \left(1-\rho^2\right)\int_0^{\infty}\frac{dF_W(w)}{dw}\times\left(\Phi^{-1}\left(F_W(w)\right)\right)^2\,dw \\
&= \left(1-\rho^2\right)\int_0^{1}\frac{dF_W\left(F_W^{-1}(s)\right)}{dw}\times\left(\Phi^{-1}(s)\right)^2\,dF_W^{-1}(s) \\
&= \left(1-\rho^2\right)\int_0^{1}\frac{dF_W\left(F_W^{-1}(s)\right)}{dw}\times\left(\Phi^{-1}(s)\right)^2\frac{1}{dF_W\left(F_W^{-1}(s)\right)/dw}\,ds \\
&= \left(1-\rho^2\right)\int_0^{1}\left(\Phi^{-1}(s)\right)^2\,ds \\
&= \left(1-\rho^2\right)\int_{-\infty}^{\infty}s^2\,d\Phi(s) = \left(1-\rho^2\right)\int_{-\infty}^{\infty}s^2\phi(s)\,ds = 1-\rho^2 \qquad (276)
\end{aligned}$$

Next

$$\begin{aligned}
E\left(\frac{\beta}{\gamma}-W^{\beta}\beta\gamma^{\beta-1}\right) &= \int_0^{\infty}\left(\frac{\beta}{\gamma}-w^{\beta}\beta\gamma^{\beta-1}\right)\gamma^{\beta}\beta w^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\,dw \\
&= \frac{\beta}{\gamma}+\left(w^{\beta}\beta\gamma^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\Big|_0^{\infty}-\int_0^{\infty}\beta^2(\gamma w)^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\,dw\right) \\
&= \frac{\beta}{\gamma}-\frac{\beta}{\gamma}\int_0^{\infty}\gamma^{\beta}\beta w^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\,dw = 0 \qquad (277)
\end{aligned}$$

Next, we want to show that

$$E\left(\log(\gamma)+1/\beta+\log(W)-(\gamma W)^{\beta}\log(\gamma W)\right) = 0 \tag{278}$$

We have

$$\int_0^{\infty}-(\gamma w)^{\beta}\log(\gamma)\left(\gamma^{\beta}\beta w^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\right)\,dw$$

$$\begin{aligned}
&= (\gamma w)^{\beta}\log(\gamma)\exp\left(-(\gamma w)^{\beta}\right)\Big|_0^{\infty}-\log(\gamma)\int_0^{\infty}\gamma^{\beta}\beta w^{\beta-1}\exp\left(-(\gamma w)^{\beta}\right)\,dw \\
&= -\log(\gamma) \qquad (279)
\end{aligned}$$

Also,

$$\int_0^\infty -(\gamma w)^\beta \log(w)\left(\gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\right)\,dw$$

$$\begin{aligned}
&= (\gamma w)^\beta \log(w)\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty - \int_0^\infty \frac{d}{dw}\left((\gamma w)^\beta \log(w)\right)\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= -\int_0^\infty \left(\gamma^\beta \beta w^{\beta-1}\log(w) + \gamma^\beta w^{\beta-1}\right)\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= -\int_0^\infty \log(w)\gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw - \int_0^\infty \gamma^\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= -\int_0^\infty \log(w)\gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw - 1/\beta \qquad (280)
\end{aligned}$$

Results (279) and (280) imply that

$$\begin{aligned}
E\left(\log(\gamma) + 1/\beta + \log(W) - (\gamma W)^\beta \log(\gamma W)\right) &= \log(\gamma) + 1/\beta + E\left(\log(W)\right) \qquad (281) \\
&\quad - E\left((\gamma W)^\beta \log(\gamma)\right) - E\left((\gamma W)^\beta \log(w)\right) \\
&= \log(\gamma) + 1/\beta + E\left(\log(W)\right) \\
&\quad - \log(\gamma) - E\left(\log(W)\right) - 1/\beta = 0
\end{aligned}$$

as desired.

Next,

$$\int_0^\infty \left(\frac{\beta}{\gamma^2} + w^\beta \beta(\beta-1)\gamma^{\beta-2}\right)\gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw$$

$$\begin{aligned}
&= \frac{\beta}{\gamma^2} + \left(-w^\beta \beta(\beta-1)\gamma^{\beta-2}\exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \int_0^\infty \beta w^{\beta-1}\beta(\beta-1)\gamma^{\beta-2}\exp\left(-(\gamma w)^\beta\right)\,dw\right) \\
&= \frac{\beta}{\gamma^2} + \frac{\beta(\beta-1)}{\gamma^2}\int_0^\infty \gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&= \frac{\beta^2}{\gamma^2} \qquad (282)
\end{aligned}$$

Next,

$$\int_0^\infty (\gamma w)^\beta \left(\log(\gamma w)\right)^2 \gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw$$

$$\begin{aligned}
&= \int_0^\infty \gamma^\beta w^\beta \left(\log(w)\right)^2 \gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\quad + \int_0^\infty \gamma^\beta w^\beta 2\log(\gamma)\log(w)\gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\quad + \int_0^\infty \gamma^\beta w^\beta \left(\log(\gamma)\right)^2 \gamma^\beta \beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\
&\equiv T_1 + T_2 + T_3 \qquad (283)
\end{aligned}$$

We have

$$
\begin{aligned}
T_1 &= -\gamma^\beta w^\beta (\log(w))^2 \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty \\
&\quad + \int_0^\infty \gamma^\beta \left(\beta w^{\beta-1}(\log(w))^2 + 2\log(w) w^{\beta-1}\right) \exp\left(-(\gamma w)^\beta\right)\, dw \\
&= E\left((\log(w))^2\right) + \frac{2}{\beta} E\left(\log(w)\right) \qquad (284)
\end{aligned}
$$

and

$$
\begin{aligned}
T_2 &= -2\log(\gamma)\gamma^\beta w^\beta \log(w) \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty \\
&\quad + 2\log(\gamma) \int_0^\infty \gamma^\beta \left(\beta w^{\beta-1}\log(w) + w^{\beta-1}\right) \exp\left(-(\gamma w)^\beta\right)\, dw \\
&= 2\log(\gamma) E\left(\log(w)\right) + \frac{2\log(\gamma)}{\beta} \qquad (285)
\end{aligned}
$$

and

$$
\begin{aligned}
T_3 &= -\left(\log(\gamma)\right)^2 \gamma^\beta w^\beta \exp\left(-(\gamma w)^\beta\right)\Big|_0^\infty + \left(\log(\gamma)\right)^2 \int_0^\infty \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw \\
&= \left(\log(\gamma)\right)^2 \qquad (286)
\end{aligned}
$$

Results (283)–(286) yield

$$
\begin{aligned}
\int_0^\infty (\gamma w)^\beta \left(\log(\gamma w)\right)^2 \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right)\, dw &= E\left((\log(w))^2\right) + \frac{2}{\beta} E\left(\log(w)\right) \\
&\quad + 2\log(\gamma) E\left(\log(w)\right) + \frac{2\log(\gamma)}{\beta} \\
&\quad + \left(\log(\gamma)\right)^2 \qquad (287)
\end{aligned}
$$

16 Appendix G—Three bivariate normal results

Assume that

$$
\begin{pmatrix} S \\ T \end{pmatrix} \sim \mathrm{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)
$$

Then

$$
\begin{aligned}
E\left(ST^2\right) &= \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{2\pi}} \exp\left(-t^2/2\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{s}{\sqrt{1-\rho^2}} \exp\left(-(s-\rho t)^2/(2(1-\rho^2))\right)\, ds\, dt \\
&= \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{2\pi}} \exp\left(-t^2/2\right) \rho t\, dt = 0 \qquad (288)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
E\left(S^2T^2\right) &= \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{2\pi}} \exp\left(-t^2/2\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{s^2}{\sqrt{1-\rho^2}} \exp\left(-(s-\rho t)^2/(2(1-\rho^2))\right)\, ds\, dt \\
&= \int_{-\infty}^{\infty} \frac{t^2}{\sqrt{2\pi}} \exp\left(-t^2/2\right) \left(1-\rho^2+\rho^2 t^2\right) dt = 1 + 2\rho^2 \qquad (289)
\end{aligned}
$$

and

$$
\begin{aligned}
E\left(ST^3\right) &= \int_{-\infty}^{\infty} \frac{t^3}{\sqrt{2\pi}} \exp\left(-t^2/2\right) \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{s}{\sqrt{1-\rho^2}} \exp\left(-(s-\rho t)^2/(2(1-\rho^2))\right) \, ds \, dt \\
&= \int_{-\infty}^{\infty} \frac{t^3}{\sqrt{2\pi}} \exp\left(-t^2/2\right) \rho t \, dt = 3\rho \qquad (290)
\end{aligned}
$$

17 Appendix H—Existence (and finiteness) of the expectations involving partial derivatives of y

To complete the calculations presented in Appendices E1-E3, we need to prove that a number of integrals involving y and its partial derivatives are well-defined. The "proofs" are all similar and are related to arguments made in the proof of Theorem 2. Let w_{low} and w_{up} be defined as in that proof. (We assume that results (40) and (41) hold.)

First, consider $E(\frac{\partial y}{\partial \gamma})$. From result (137) we have

$$
\frac{\partial y}{\partial \gamma} = \frac{\exp\left(-(\gamma w)^\beta\right) \beta \gamma^{\beta-1} w^\beta}{\phi(y)} \qquad (291)
$$

where

$$
y \equiv \Phi^{-1}\left(1 - \exp(-(\gamma w)^\beta\right)
$$

Our claim, which in essence establishes all the results in this Appendix, is that

$$
\int_0^\infty \left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)} \right) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \, dw
$$

is well-defined and finite.

We have

$$
\begin{aligned}
& \int_0^\infty \left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)} \right) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \, dw \\
= & \int_0^{w_{\text{low}}} \left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)} \right) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \, dw \\
& + \int_{w_{\text{low}}}^{w_{\text{up}}} \left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)} \right) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \, dw \\
& + \int_{w_{\text{up}}}^{\infty} \left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)} \right) \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \, dw \\
\equiv & \; T_1 + T_2 + T_3 \qquad (292)
\end{aligned}
$$

It is clear that T_2 is well-defined and finite.

Now consider T_3. By result (41), for $w > w_{\text{up}}$

$$
\begin{aligned}
(\gamma w)^\beta &> (\gamma w_{\text{up}})^\beta > (\gamma_{\text{low}} w_{\text{up}})^\beta \\
&> (\gamma_{\text{low}} w_{\text{up}})^{\beta_{\text{low}}} > (\gamma w_{3/4})^\beta
\end{aligned}
$$

so

$$1-\exp\left(-(\gamma w)^\beta\right) > 1-\exp\left(-(\gamma w_{3/4})^\beta\right) = 3/4 \tag{293}$$

By Lemma 1 and (293), we have

$$T_3 \quad < \quad \int_{w_{\rm up}}^{\infty} \frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\exp\left(-(\gamma w)^\beta\right)\times\Phi^{-1}(3/4)}\,\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \tag{294}$$

which is clearly well-defined and finite.

Now consider T_1. By result (40), for $w < w_{\rm low}$

$$\begin{aligned}(\gamma w)^\beta &< (\gamma w_{\rm low})^\beta < (\gamma_{\rm up} w_{\rm low})^\beta \\ &< (\gamma_{\rm up} w_{\rm low})^{\beta_{\rm low}} < \min\left(1/4, (\gamma w_{1/4})^\beta\right)\end{aligned} \tag{295}$$

so

$$1-\exp\left(-(\gamma w)^\beta\right) < 1-\exp\left(-(\gamma w_{1/4})^\beta\right) = 1/4 \tag{296}$$

By Lemma 1 and (296), we have

$$T_1 \quad < \quad \int_0^{w_{\rm low}} \frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\left(1-\exp\left(-(\gamma w)^\beta\right)\right)\times\left(-\Phi^{-1}(1/4)\right)}\,\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \tag{297}$$

Now for $w < w_{\rm low}$, by (40) we have (as in result (295))

$$(\gamma w)^\beta < (\gamma_{\rm up} w_{\rm low})^{\beta_{\rm low}} < 1/4 \tag{298}$$

Thus, for $w < w_{\rm low}$ and $x \equiv (\gamma w)^\beta$,

$$\begin{aligned}1-\exp(-(\gamma w)^\beta) &= 1-\left(1-x+x^2/2!-x^3/3!+x^4/4!-\ldots\right) \\ &= x\left(1-x/2!+x^2/3!-x^3/4!+\ldots\right) \\ &> x(1-x/2) \\ &> x(1-1/8) = x\times 7/8\end{aligned} \tag{299}$$

Results (297) and (299) imply

$$T_1 \quad < \quad \int_0^{w_{\rm low}} \frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\left((\gamma w)^\beta\times 7/8\right)\times\left(-\Phi^{-1}(1/4)\right)}\,\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \tag{300}$$

which is clearly well-defined and finite. Results (291), (292), (294), and (300) imply that $E\left(\frac{\partial y}{\partial\gamma}\right)$ is well-defined and finite.

Now consider $E(y\frac{\partial y}{\partial \gamma})$. We have

$$\begin{aligned}&\int_0^{\infty}\left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)}\right)|y|\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\ =\ &\int_0^{w_{\rm low}}\left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)}\right)|y|\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\ &+\int_{w_{\rm low}}^{w_{\rm up}}\left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)}\right)|y|\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\ &+\int_{w_{\rm up}}^{\infty}\left(\frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\phi(y)}\right)|y|\gamma^\beta\beta w^{\beta-1}\exp\left(-(\gamma w)^\beta\right)\,dw \\ \equiv\ & T_1+T_2+T_3\end{aligned} \tag{301}$$

It is clear that T_2 is well-defined and finite.

Now consider T_3. By Lemma 1, we have

$$T_3 \quad < \quad \int_{w_{\text{up}}}^{\infty} \frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\exp\left(-(\gamma w)^\beta\right) \times y} \, y\gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \, dw \tag{302}$$

which is clearly well-defined and finite.

Now consider T_1. By Lemma 1, we have

$$T_1 \quad < \quad \int_{0}^{w_{\text{low}}} \frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\left(1-\exp\left(-(\gamma w)^\beta\right)\right) \times (-y)} \, (-y)\gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \, dw \tag{303}$$

Results (303) and (299) imply

$$T_1 \quad < \quad \int_{0}^{w_{\text{low}}} \frac{w^\beta \exp\left(-(\gamma w)^\beta\right)}{\left((\gamma w)^\beta \times 7/8\right)} \, \gamma^\beta \beta w^{\beta-1} \exp\left(-(\gamma w)^\beta\right) \, dw \tag{304}$$

which is clearly well-defined and finite. Results (301), (302) and (304) imply that $E\left(y\frac{\partial y}{\partial \gamma}\right)$ is well-defined and finite.

It is clear via an inspection of the remaining partial derivatives in Appendix D that their expectations can be handled similarly. (We also use the additional fact that for $\beta > 1$, $\log(w)w^{\beta-1}$ converges to 0 as w decreases to 0.)

18 Appendix I—Positive definite information matrix

To invoke Lehmann's Theorem 4.2, we need to establish that the information matrix is positive definite. In Appendices E2 and E3 we establish that

$$E\left(-\frac{\partial^2 \log(f(x,w))}{\partial\theta_i \partial\theta_j}\right) = E\left(\frac{\partial f/\partial\theta_i}{f} \times \frac{\partial f/\partial\theta_j}{f}\right) \tag{305}$$

Thus

$$\begin{aligned} \mathbf{a}^T \mathbf{I}(\theta)\mathbf{a} \quad &= \quad \sum_{i=1}^{5}\sum_{j=1}^{5} a_i a_j E\left(\frac{\partial f/\partial\theta_i}{f} \times \frac{\partial f/\partial\theta_j}{f}\right) \\ &= \quad E\left(\left(\sum_{i=1}^{5} a_i \frac{\partial f/\partial\theta_i}{f}\right)^2\right) \geq 0 \end{aligned} \tag{306}$$

To complete the proof that $\mathbf{I}(\theta)$ is positive definite we need to show that

$$\sum_{i=1}^{5} a_i \frac{\partial f/\partial\theta_i}{f} = 0 \text{ a.e.} \tag{307}$$

implies $\mathbf{a} = \mathbf{0}$. From result (172) we have

$$\begin{aligned}
\sum_{i=1}^{5} a_i \frac{\partial f/\partial\theta_i}{f} &= a_1 \times \left(\frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\right) \qquad (308)\\
&+ a_2 \times \left(\frac{-1}{\sigma} + \frac{1}{\sigma}\frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)}{1-\rho^2}\left(\frac{x-\mu}{\sigma}\right)\right)\\
&+ a_3 \times \left(\frac{\rho}{1-\rho^2} + \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right) y}{1-\rho^2} - \frac{\left(\frac{x-\mu}{\sigma} - \rho y\right)^2 \rho}{(1-\rho^2)^2}\right)\\
&+ a_4 \times \left(\frac{\beta}{\gamma} - w^\beta \beta \gamma^{\beta-1} + \left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \gamma}\right)\\
&+ a_5 \times \left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w) + \left(\frac{x-\mu}{\sigma} - \rho y\right)\frac{\rho}{1-\rho^2}\frac{\partial y}{\partial \beta}\right)
\end{aligned}$$

From result (137), we have

$$\frac{\partial y}{\partial \gamma} = \beta\gamma^{\beta-1} \times w^\beta \times \exp\left(-(\gamma w)^\beta\right)/\phi(y) \qquad (309)$$

From result (138), we have

$$\frac{\partial y}{\partial \beta} = (\gamma w)^\beta \log(\gamma w) \times \exp\left(-(\gamma w)^\beta\right)/\phi(y) \qquad (310)$$

Recall that

$$y \equiv \Phi^{-1}\left(1 - \exp(-(\gamma w)^\beta\right)$$

Now let $\epsilon > 0$ be given. Then results (307) through (310) imply that given any w_0, we can find an associated x, w rectangle chosen so that $(x-\mu)/\sigma - \rho y$ is small in the rectangle such that

$$\begin{aligned}
\Bigg| a_2 \times \left(\frac{-1}{\sigma}\right) &+ a_3 \times \left(\frac{\rho}{1-\rho^2}\right) \qquad (311)\\
&+ a_4 \times \left(\frac{\beta}{\gamma} - w^\beta\beta\gamma^{\beta-1}\right)\\
&+ a_5 \times \left(\log\gamma + \frac{1}{\beta} + \log(w) - (\gamma w)^\beta \log(\gamma w)\right)\Bigg| < \epsilon/2
\end{aligned}$$

for some (x,w) in the rectangle.

A suitable rectangle can be written as $[x_0 - \delta, x_0 + \delta] \times [w_0 - \delta, w_0 + \delta]$ where δ can be made arbitrarily small, $(x_0 - \mu)/\sigma - \rho y_0 = 0$, and $y_0 = \Phi^{-1}\left(1 - \exp(-(\gamma w_0)^\beta\right)$. By (307), there must be some (x,w) in the rectangle for which $\sum_{i=1}^{5} a_i \frac{\partial f/\partial\theta_i}{f} = 0$.

Taking w_0 large enough

$$|a_4 + a_5 K \log(\gamma w)| < \epsilon \qquad (312)$$

for K fixed and positive and w arbitrarily large. As ϵ was arbitrary, this implies that a_4 and a_5 equal 0.

Now, given results (307) and (308) and the fact that $a_4 = a_5 = 0$, given any $\epsilon > 0$, we can find an x, w region of positive measure (chosen so that y is large and $(x-\mu)/\sigma$ is bounded) such that (taking y large enough)

$$\left| a_3 \times \left(\frac{-\rho}{1-\rho^2} - \frac{\rho^3}{(1-\rho^2)^2}\right)\right| < \epsilon \qquad (313)$$

This implies that $a_3 = 0$ or $\rho = 0$. If $\rho = 0$, then (given that $a_4 = a_5 = 0$)

$$\sum_{i=1}^{5} a_i \frac{\partial f/\partial\theta_i}{f} = a_1 \times \left(\frac{1}{\sigma}\frac{x-\mu}{\sigma}\right) + a_2 \times \left(\frac{-1}{\sigma} + \frac{1}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^2\right) + a_3 \times \left(\frac{x-\mu}{\sigma} y\right) \tag{314}$$

Given results (307) and (314), given any $\epsilon > 0$, we can find an x, w region of positive measure (chosen so that y is large and $(x-\mu)/\sigma$ is bounded above and bounded below away from 0) such that (taking y large enough)

$$\left| a_3 \left(\frac{x-\mu}{\sigma}\right)\right| < \epsilon$$

for arbitrary $(x-\mu)/\sigma$ in the bounded region. Thus, $a_3 = 0$.

Next, given results (307) and (308) and the fact that $a_3 = a_4 = a_5 = 0$, given any $\epsilon > 0$, we can find an x, w region of positive measure (chosen so that $(x-\mu)/\sigma$ is large and y is bounded) such that (letting x get large enough)

$$\left| a_2 \times \frac{1}{\sigma(1-\rho^2)}\right| < \epsilon \tag{315}$$

This implies that $a_2 = 0$.

Finally, results (307) and (308) and the fact that $a_2 = a_3 = a_4 = a_5 = 0$ imply that $a_1 = 0$, or $\mathbf{a} = 0$ as needed.

19 Appendix J—Lehmann's condition D

Lehmann's condition D requires that there exists an open 5-dimensional set, S, that contains the point $\boldsymbol{\theta}_0 = (\gamma_0, \beta_0, \rho_0, \mu_0, \sigma_0)$, and that there exist functions $M_{ijk}(x, w)$ such that

$$\left|\frac{\partial^3 \log(f(x, w; \boldsymbol{\theta}))}{\partial\theta_i\theta_j\theta_k}\right| \leq M_{ijk}(x, w) \tag{316}$$

for all $\boldsymbol{\theta} \in S$, and

$$E_{\boldsymbol{\theta}_0}\left(M_{ijk}(x, w)\right) < \infty \tag{317}$$

for all i, j, k.

The third partials of $\log(f(x, w; \boldsymbol{\theta}))$ are displayed in Appendix C. There are 35 third partials. They can all be handled with five techniques. We will present these techniques in detail by using them to handle some of the 35 partials, and then describe which of them are needed for each of the remaining 35 third partials.

Let $0 < \gamma_{\text{low}} < \gamma_0 < \gamma_{\text{up}}$, $\gamma_{\text{low}} < 1 < \gamma_{\text{up}}$, $1 < \beta_{\text{low}} < \beta_0 < \beta_{\text{up}}$, $-1 < \rho_{\text{low}} < \rho_0 < \rho_{\text{up}} < 1$, $\mu_{\text{low}} < \mu_0 < \mu_{\text{up}}$, and $0 < \sigma_{\text{low}} < \sigma_0 < \sigma_{\text{up}}$. Then

$$S \equiv (\gamma_{\text{low}}, \gamma_{\text{up}}) \times (\beta_{\text{low}}, \beta_{\text{up}}) \times (\rho_{\text{low}}, \rho_{\text{up}}) \times (\mu_{\text{low}}, \mu_{\text{up}}) \times (\sigma_{\text{low}}, \sigma_{\text{up}})$$

is an open neighborhood of $\boldsymbol{\theta}_0 = (\gamma_0, \beta_0, \rho_0, \mu_0, \sigma_0)$. Let

$$\bar{S} \equiv [\gamma_{\text{low}}, \gamma_{\text{up}}] \times [\beta_{\text{low}}, \beta_{\text{up}}] \times [\rho_{\text{low}}, \rho_{\text{up}}] \times [\mu_{\text{low}}, \mu_{\text{up}}] \times [\sigma_{\text{low}}, \sigma_{\text{up}}]$$

denote the closure of S.

Technique 1—portion of a partial is a continuous function of the parameters alone (not involving w or x)

Because $\bar{S}$ is compact, the image of any continuous function on $\bar{S}$ is bounded. Thus, in those cases in which a portion of the third partial is simply a function of $\gamma, \beta, \rho, \mu, \sigma$ (and not of x or w) we can find a constant M that dominates the absolute value of that portion of the partial for all elements of S.

This technique permits us to fully handle third partials (listed in Appendix C) 1, 6–9, and to handle those portions of the other third partials that are dependent only on the parameters (and not on x or w).

Technique 2—portion of a partial is $\left(w^\beta \times \exp\left(-(\gamma w)^\beta\right) \times y \times \exp\left(y^2/2\right)\right)^k$

We proceed here in a fashion similar to the fashion in which we proceeded in Appendix H. Let w_{low} and w_{up} be defined as in the proof of Theorem 2. That is, we assume that results (36)–(41) hold with γ, β replaced by γ_0, β_0. We have

$$\int_0^\infty \left|\frac{w^\beta \exp\left(-(\gamma w)^\beta\right) y}{\phi(y)}\right|^k \gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right) \, dw$$

$$\begin{aligned}
&= \int_0^{w_{\text{low}}} \left|\frac{w^\beta \exp\left(-(\gamma w)^\beta\right) y}{\phi(y)}\right|^k \gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right) \, dw \\
&\quad + \int_{w_{\text{low}}}^{w_{\text{up}}} \left|\frac{w^\beta \exp\left(-(\gamma w)^\beta\right) y}{\phi(y)}\right|^k \gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right) \, dw \\
&\quad + \int_{w_{\text{up}}}^{\infty} \left|\frac{w^\beta \exp\left(-(\gamma w)^\beta\right) y}{\phi(y)}\right|^k \gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right) \, dw \\
&\equiv T_1 + T_2 + T_3
\end{aligned} \tag{318}$$

Note that we are now drawing a distinction between general $(\gamma, \beta) \in [\gamma_{\text{low}}, \gamma_{\text{up}}] \times [\beta_{\text{low}}, \beta_{\text{up}}]$ and the true parmaeter vector (γ_0, β_0).

It is clear that for $(\gamma, \beta, \rho, \mu, \sigma) \in \bar{S}$, the $\left|w^\beta \exp\left(-(\gamma w)^\beta\right) y/\phi(y)\right|^k$ in T_2 can be dominated by a constant that does not depend on the parameter values, and that the resulting integral over $(w_{\text{low}}, w_{\text{up}})$ is finite.

Now consider T_3. By Lemma 1 and the facts that $w_{\text{up}} > 1$ and for $w > w_{\text{up}}$,

$$(\gamma w)^\beta > (\gamma_{\text{low}} w_{\text{up}})^\beta > (\gamma_{\text{low}} w_{\text{up}})^{\beta_{\text{low}}} > (\gamma_0 w_{3/4})^{\beta_0}$$

(assumption (41)), for $w > w_{\text{up}}$ we have

$$\begin{aligned}
\left|\frac{w^\beta \exp\left(-(\gamma w)^\beta\right) y}{\phi(y)}\right|^k &< \left|\frac{w^\beta \times \exp\left(-(\gamma w)^\beta\right) \times y}{\exp\left(-(\gamma w)^\beta\right) \times y}\right|^k \\
&< w^{\beta_{\text{up}} k}
\end{aligned} \tag{319}$$

Clearly, $w^{\beta_{\text{up}} k}$ does not depend upon the parameter values and the resulting integral over (w_{up}, ∞) is finite.

Now consider T_1. By Lemma 1, (299), and the facts that $\gamma_{\text{low}} < 1$ and

$$(\gamma w)^\beta < (\gamma_{\text{up}} w_{\text{low}})^\beta < (\gamma_{\text{up}} w_{\text{low}})^{\beta_{\text{low}}} < \min(1/4, (\gamma_0 w_{1/4})^{\beta_0})$$

we have

$$\begin{aligned} \left| \frac{w^\beta \exp\left(-(\gamma w)^\beta\right) y}{\phi(y)} \right|^k &< \left| \frac{w^\beta \times \exp\left(-(\gamma w)^\beta\right) \times y}{(1 - \exp\left(-(\gamma w)^\beta\right)) \times y} \right|^k \\ &< \left(\frac{w^\beta}{(\gamma w)^\beta \times 7/8} \right)^k < \left(\frac{1}{\gamma_{\text{low}}^{\beta_{\text{up}}} \times 7/8} \right)^k \end{aligned} \tag{320}$$

Clearly, this quantity does not depend upon the parameter values and the resulting integral over $(-\infty, w_{\text{low}})$ is finite.

This analysis of T_1, T_2, and T_3 establishes that on S,

$$\left(w^\beta \times \exp\left(-(\gamma w)^\beta \right) \times y \times \exp\left(y^2/2\right) \right)^k$$

can be dominated by a function that does not depend on the parameter values whose expectation (with respect to the $\boldsymbol{\theta}_0$ pdf) is finite.

Techniques 1, 2, and 5 (see below) together permit us to fully handle third partials 18, 22, and 27–31.

Technique 3—handling $(x-\mu)/\sigma$ and $((x-\mu)/\sigma)^2$

First note that $|x-\mu|/\sigma < ((x-\mu)/\sigma)^2 + 1$ so we only need to address $((x-\mu)/\sigma)^2$.

Define

$$M \equiv \max\left(|\mu_{\text{low}}|, |\mu_{\text{up}}|\right)$$

Then for $\boldsymbol{\theta} \in S$

$$\left(\frac{x-\mu}{\sigma} \right)^2 \leq \frac{(|x| + M)^2}{\sigma_{\text{low}}^2} = \frac{|x|^2 + 2|x|M + M^2}{\sigma_{\text{low}}^2} \tag{321}$$

Define

$$y_0 \equiv \Phi^{-1}\left(1 - \exp\left(-(\gamma_0 w)^{\beta_0}\right)\right)$$

Then we have

$$\begin{aligned} &\int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_0\sqrt{1-\rho_0^2}} \exp\left(-\left((x-\mu_0)/\sigma_0 - \rho_0 y_0\right)^2 / (2(1-\rho_0^2)) \right) dx \\ = &\int_{-\infty}^{\infty} |x+\mu_0| \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_0\sqrt{1-\rho_0^2}} \exp\left(-\left(x/\left(\sigma_0\sqrt{1-\rho_0^2}\right) - \rho_0 y_0/\sqrt{1-\rho_0^2} \right)^2 /2 \right) dx \\ = &\int_{-\infty}^{\infty} \left| \sigma_0\sqrt{1-\rho_0^2}\, x + \mu_0 \right| \frac{1}{\sqrt{2\pi}} \exp\left(-\left(x - \rho_0 y_0/\sqrt{1-\rho_0^2} \right)^2 /2 \right) dx \\ = &\int_{-\infty}^{\infty} \left| \sigma_0\sqrt{1-\rho_0^2} \left(x + \rho_0 y_0/\sqrt{1-\rho_0^2} \right) + \mu_0 \right| \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) dx \\ = &\int_{-\infty}^{\infty} \left| \sigma_0\sqrt{1-\rho_0^2}\, x + \sigma_0\rho_0 y_0 + \mu_0 \right| \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right) dx \\ \leq &\ \sigma_0\sqrt{1-\rho_0^2} \frac{2}{\sqrt{2\pi}} + \sigma_0|\rho_0||y_0| + |\mu_0| \end{aligned} \tag{322}$$

By a similar argument,

$$\int_{-\infty}^{\infty} |x|^2 \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_0\sqrt{1-\rho_0^2}} \exp\left(- \left((x-\mu_0)/\sigma_0 - \rho_0 y_0\right)^2 / (2(1-\rho_0^2)) \right) dx$$

$$= \sigma_0^2(1-\rho_0^2) + \sigma_0^2\rho_0^2 y_0^2 + 2\mu_0\sigma_0\rho_0 y_0 + \mu_0^2 \qquad (323)$$

Results (321)–(323) and technique 4 below establish that for $\boldsymbol{\theta} \in S$, $((x-\mu)/\sigma)^2$ (and thus $|x-\mu|/\sigma$) can be dominated by a function that does not depend on $\boldsymbol{\theta}$ and that is integrable with respect to $f(x, w; \boldsymbol{\theta}_0)$.

Technique 4—handling y and y^2

Because $|y| < y^2 + 1$, we can focus on y^2.

Let w_{low} and w_{up} be defined as in the proof of Theorem 2. That is, we assume that results (37)–(41) hold with γ, β replaced by γ_0, β_0. We have

$$\begin{aligned} \int_0^{\infty} y^2 \gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right) dw &= \int_0^{w_{\text{low}}} y^2 \gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right) dw \\ &\quad + \int_{w_{\text{low}}}^{w_{\text{up}}} y^2 \gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right) dw \\ &\quad + \int_{w_{\text{up}}}^{\infty} y^2 \gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right) dw \\ &\equiv T_1 + T_2 + T_3 \end{aligned} \qquad (324)$$

It is clear that for $(\gamma, \beta, \rho, \mu, \sigma) \in \bar{S}$, $y^2 = \left(\Phi^{-1}(1 - \exp(-(\gamma w)^{\beta}))\right)^2$ in T_2 can be dominated by a constant that does not depend on the parameter values, and that the resulting T_2 integral is finite.

Now consider T_3.

We claim that for w large enough, y^2 is "like" $2(\gamma w)^{\beta}$. From (39) and (41) (with γ, β replaced by γ_0, β_0), we know that for $w > w_{\text{up}}$ and $\boldsymbol{\theta} \in S$,

$$(\gamma w)^{\beta} > (\gamma_{\text{low}} w_{\text{up}})^{\beta} > (\gamma_{\text{low}} w_{\text{up}})^{\beta_{\text{low}}} > (\gamma_0 w_{3/4})^{\beta_0} \qquad (325)$$

so

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) > \Phi^{-1}(3/4) > 0$$

Thus, we can apply Lemma 1 to obtain

$$\frac{y^2}{1+y^2} \frac{1}{\sqrt{2\pi}} \exp\left(-y^2/2\right) < \exp\left(-(\gamma w)^{\beta}\right) y < \frac{1}{\sqrt{2\pi}} \exp\left(-y^2/2\right) \qquad (326)$$

Taking logs, we then obtain

$$\log\left(\frac{y^2}{1+y^2}\right) + \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{y^2}{2} < -(\gamma w)^{\beta} + \log(y) < \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{y^2}{2} \qquad (327)$$

or

$$\begin{aligned} \log\left(\frac{y^2}{1+y^2}\right)/(-y^2/2) + \log\left(\frac{1}{\sqrt{2\pi}}\right)/(-y^2/2) + 1 &> (\gamma w)^{\beta}/(y^2/2) + \log(y)/(-y^2/2) \\ &> \log\left(\frac{1}{\sqrt{2\pi}}\right)/(-y^2/2) + 1 \end{aligned} \qquad (328)$$

By (325), for $w > w_{\text{up}}$ and any $\boldsymbol{\theta} \in S$,

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) > \Phi^{-1}\left(1 - \exp\left(-(\gamma_{\text{low}} w_{\text{up}})^{\beta_{\text{low}}}\right)\right)$$

so we can make y arbitrarily large by choosing w_{up} large enough. Thus, by result (328), given any $\delta > 0$, we can choose w_{up} large enough so that $w > w_{\text{up}}$ implies

$$1 - \delta < (\gamma w)^{\beta}/(y^2/2) < 1 + \delta$$

or, taking $\delta = 1/2$,

$$y^2/4 < (\gamma w)^{\beta}$$

or

$$y^2 < 4(\gamma w)^{\beta} < 4(\gamma_{\text{up}} w)^{\beta_{\text{up}}} \tag{329}$$

Thus, for $(\gamma, \beta, \rho, \mu, \sigma) \in S$, the y^2 in T_3 can be dominated by a function that does not depend on the parameter values, and the resulting integral over (w_{up}, ∞) is finite.

Now consider T_1.

We claim that for w small enough, y^2 is "like" $-2\beta \log(\gamma w)$. From (38) and (40) (with γ, β replaced by γ_0, β_0), we know that for $w < w_{\text{low}}$ and $\boldsymbol{\theta} \in S$,

$$(\gamma w)^{\beta} < (\gamma_{\text{up}} w_{\text{low}})^{\beta_{\text{low}}} < \min\left(1/4, (\gamma_0 w_{1/4})^{\beta_0}\right) \tag{330}$$

so

$$y = \Phi^{-1}(1 - \exp\left(-(\gamma w)^{\beta}\right) < \Phi^{-1}(1/4) < 0$$

Thus, we can apply Lemma 1 to obtain

$$\frac{y^2}{1+y^2}\frac{1}{\sqrt{2\pi}}\exp\left(-y^2/2\right) < \left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right)(-y) < \frac{1}{\sqrt{2\pi}}\exp\left(-y^2/2\right) \tag{331}$$

Taking logs, we then obtain

$$\log\left(\frac{y^2}{1+y^2}\right) + \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{y^2}{2} < \log\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) + \log(-y) < \log\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{y^2}{2} \tag{332}$$

or

$$\begin{aligned} \log\left(\frac{y^2}{1+y^2}\right)/(-y^2/2) + \log\left(\frac{1}{\sqrt{2\pi}}\right)/(-y^2/2) + 1 \quad &> \quad -\log\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right)/(y^2/2) \\ &\qquad + \log(-y)/(-y^2/2) \\ &> \quad \log\left(\frac{1}{\sqrt{2\pi}}\right)/(-y^2/2) + 1 \end{aligned} \tag{333}$$

Because $\gamma_{\text{up}} w_{\text{low}} < 1$ (result (40)), for $w < w_{\text{low}}$ and and $\boldsymbol{\theta} \in S$,

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) < \Phi^{-1}\left(1 - \exp\left(-(\gamma_{\text{up}} w_{\text{low}})^{\beta_{\text{low}}}\right)\right)$$

so we can make $-y$ arbitrarily large by choosing w_{low} small enough. Thus, by result (333), given any $\delta > 0$, we can choose w_{low} small enough so that $w < w_{\text{low}}$ implies

$$1 - \delta < -\log\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right)/(y^2/2) < 1 + \delta \tag{334}$$

Now by results (299) and (330), for $w < w_{\text{low}}$ and $\boldsymbol{\theta} \in S$,

$$1 - \exp\left(-(\gamma w)^{\beta}\right) > (\gamma w)^{\beta} \times 7/8$$

so

$$\log\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) > \beta \log(\gamma w) + \log(7/8)$$

or

$$-\log\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right)/(y^2/2) < -\beta \log(\gamma w)/(y^2/2) - \log(7/8)/(y^2/2) \tag{335}$$

Combining results (334) and (335) and the fact that $-y$ can be made arbitrarily large by making w_{low} small enough, we know that we can choose w_{low} small enough so that for $w < w_{\text{low}}$ and $\boldsymbol{\theta} \in S$

$$y^2/4 < -\beta \log(\gamma w) \tag{336}$$

By assumption (40), for $w < w_{\text{low}}$ and $\boldsymbol{\theta} \in S$

$$1 > \gamma_{\text{up}} w_{\text{low}} > \gamma w > \gamma_{\text{low}} w$$

so

$$\beta \log(\gamma w) > \beta \log(\gamma_{\text{low}} w) > \beta_{\text{up}} \log(\gamma_{\text{low}} w) \tag{337}$$

By results (336) and (337), for $w < w_{\text{low}}$ (and w_{low} sufficiently small)

$$y^2 < -4\beta_{\text{up}} \log(\gamma_{\text{low}} w) \tag{338}$$

Thus, for $(\gamma, \beta, \rho, \mu, \sigma) \in S$, the y^2 in T_1 can be dominated by a function that does not depend on the parameter values, and the resulting integral over $(0, w_{\text{low}})$ is finite (because $\log(w)^r w^{\beta_0 - 1} \to 0$ as $w \to 0$).

Thus, we have established that if the parameter vector lies in S, then y^2 (and hence y) is dominated by an integrable function that does not depend on the parameter values.

Technique 5—handling $w^{k\beta}$ and $\log(w)^j$

On (w_{up}, ∞), $w^{k\beta} \log(w)^j$ is dominated by $w^{k\beta_{\text{up}}+j}$ which is integrable with respect to $\gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right)$.

For w in the compact interval $[w_{\text{low}}, w_{\text{up}}]$, and $\boldsymbol{\theta}$ in the compact region $\bar{S}$, the continuous function $w^{k\beta} \log(w)^j$ is dominated by a constant which is integrable with respect to $\gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right)$.

For $w \in (0, w_{\text{low}})$, $|w^{k\beta} \log(w)^j|$ is dominated by $|\log(w)|^j$ which is integrable with respect to $\gamma_0^{\beta_0} \beta_0 w^{\beta_0 - 1} \exp\left(-(\gamma_0 w)^{\beta_0}\right)$ because $|\log(w)|^j w^{\beta_0 - 1} \to 0$ as $w \to 0$.

Thus, if the parameter vector lies in S, then $|w^{k\beta} \log(w)^j|$ is dominated by an integrable function that does not depend on the parameter values.

Applying the five techniques

We have already used techniques 1, 2, and 5 to establish that third partials (see results (102) through (136)) 1, 6–9, 18, 22, and 27–31 satisfy Lehmann's condition D. Now we simply list the remaining partials and the techniques needed to handle them. (Note that partials of y are given in Appendix D.)

2: Techniques 1, 3, 4
3: Techniques 1, 3, 4

4: Techniques 1–5
5: Techniques 1–5
10: Techniques 1, 3, 4
11: Techniques 1, 3, 4
12: Techniques 1–4
13: Techniques 1–5
14: Techniques 1, 3, 4
15: Techniques 1, 3, 4
16: Techniques 1–4
17: Techniques 1–5
19: Techniques 1–5
20: Techniques 1–5
21: Techniques 1–5
23: Techniques 1–5
24: Techniques 1–5
25: Techniques 1–5
26: Techniques 1, 3, 4
32: Techniques 1–4
33: Techniques 1–5
34: Techniques 1–5
35: Techniques 1–5

20 Appendix K—Behavior of pseudo-truncated Weibull as $\rho \to 1$

From result (6) it is clear that we need to investigate the behavior of

$$\begin{aligned} N/D \equiv & \left(\Phi\left((c_{\mathrm{u}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}\right)-\Phi\left((c_{\mathrm{l}}-\mu)/\left(\sigma\sqrt{1-\rho^2}\right)-\rho y/\sqrt{1-\rho^2}\right)\right) \\ & \div\left(\Phi((c_{\mathrm{u}}-\mu)/\sigma)-\Phi((c_{\mathrm{l}}-\mu)/\sigma)\right) \end{aligned} \tag{339}$$

as $\rho \to 0$ where

$$y = \Phi^{-1}\left(1-\exp\left(-(\gamma w)^{\beta}\right)\right)$$

We have

$$\begin{aligned} N &= \Phi\left(\frac{c_{\mathrm{u}}-\mu}{\sigma\sqrt{1-\rho^2}}-\frac{\rho y}{\sqrt{1-\rho^2}}\right) \\ &\quad -\Phi\left(\frac{c_{\mathrm{l}}-\mu}{\sigma\sqrt{1-\rho^2}}-\frac{\rho y}{\sqrt{1-\rho^2}}\right) \\ &= \Phi\left(\frac{c_{\mathrm{u}}-\mu}{\sigma\sqrt{1-\rho^2}}-\frac{y}{\sqrt{1-\rho^2}}+\frac{(1-\rho)y}{\sqrt{1-\rho^2}}\right) \\ &\quad -\Phi\left(\frac{c_{\mathrm{l}}-\mu}{\sigma\sqrt{1-\rho^2}}-\frac{y}{\sqrt{1-\rho^2}}+\frac{(1-\rho)y}{\sqrt{1-\rho^2}}\right) \\ &\equiv \Phi\left(\arg_u\right)-\Phi\left(\arg_l\right) \end{aligned} \tag{340}$$

First note that

$$\frac{(1-\rho)y}{\sqrt{1-\rho^2}} = \frac{(1-\rho)y}{\sqrt{1-\rho}\sqrt{1+\rho}} \to 0 \tag{341}$$

as $\rho \to 1$. Next note that by the definition (7) of w_l, for $w < w_l$

$$\begin{aligned} y &= \Phi^{-1}\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) \\ &< \Phi^{-1}\left(1 - \exp\left(-(\gamma w_l)^{\beta}\right)\right) = \frac{c_l - \mu}{\sigma} \end{aligned} \tag{342}$$

Thus, from results (340)–(342), for $w < w_l$ both $\arg_l$ and $\arg_u$ converge to $+\infty$ as $\rho \to 1$, and, consequently $N \to 1 - 1 = 0$ as $\rho \to 1$.

By the definition (8) of w_u, for $w > w_u$

$$\begin{aligned} y &= \Phi^{-1}\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) \\ &> \Phi^{-1}\left(1 - \exp\left(-(\gamma w_u)^{\beta}\right)\right) = \frac{c_u - \mu}{\sigma} \end{aligned} \tag{343}$$

So from results (340), (341), and (343), for $w > w_u$ both $\arg_l$ and $\arg_u$ converge to $-\infty$ as $\rho \to 1$, and, consequently $N \to 0 - 0 = 0$ as $\rho \to 1$.

For $w \in (w_l, w_u)$

$$\begin{aligned} y &= \Phi^{-1}\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) \\ &> \Phi^{-1}\left(1 - \exp\left(-(\gamma w_l)^{\beta}\right)\right) = \frac{c_l - \mu}{\sigma} \end{aligned} \tag{344}$$

and

$$\begin{aligned} y &= \Phi^{-1}\left(1 - \exp\left(-(\gamma w)^{\beta}\right)\right) \\ &< \Phi^{-1}\left(1 - \exp\left(-(\gamma w_u)^{\beta}\right)\right) = \frac{c_u - \mu}{\sigma} \end{aligned} \tag{345}$$

So from results (340), (341), (344), and (345), for $w \in (w_l, w_u)$, $\arg_l$ converges to $-\infty$ and $\arg_u$ converges to $+\infty$ as $\rho \to 1$, and, consequently $N \to 1 - 0 = 1$ as $\rho \to 1$.

For $w = w_l$

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma w_l)^{\beta}\right)\right) = \frac{c_l - \mu}{\sigma} \tag{346}$$

From results (340), (341), and (346), for $w = w_l$, $\arg_l$ converges to 0 and $\arg_u$ converges to $+\infty$ as $\rho \to 1$, and, consequently $N \to 1 - 1/2 = 1/2$ as $\rho \to 1$.

For $w = w_u$

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma w_u)^{\beta}\right)\right) = \frac{c_u - \mu}{\sigma} \tag{347}$$

From results (340), (341), and (347), for $w = w_u$, $\arg_l$ converges to $-\infty$ as $\rho \to 1$ and $\arg_u$ converges to 0, and, consequently $N \to 1/2 - 0 = 1/2$ as $\rho \to 1$.

21 Appendix L—Proof that pseudo-truncated Weibulls are not Weibulls

We need to show that, given $\gamma_1 > 0$, $\beta_1 > 1$, there is no $\gamma_2 > 0$, $\beta_2 > 0$ such that, for all $w > 0$,

$$\begin{aligned} f_{\mathrm{PTW}}(w) &= \gamma_1^{\beta_1}\beta_1 w^{\beta_1 - 1}\exp\left(-(\gamma_1 w)^{\beta_1}\right) \\ &\quad \times \left(\Phi\left((c_{\mathrm{u}} - \mu)/\left(\sigma\sqrt{1-\rho^2}\right) - \rho y/\sqrt{1-\rho^2}\right) - \Phi\left((c_{\mathrm{l}} - \mu)/\left(\sigma\sqrt{1-\rho^2}\right) - \rho y/\sqrt{1-\rho^2}\right)\right) \\ &\quad \div \left(\Phi((c_{\mathrm{u}} - \mu)/\sigma) - \Phi((c_{\mathrm{l}} - \mu)/\sigma)\right) \\ &= \gamma_2^{\beta_2}\beta_2 w^{\beta_2 - 1}\exp\left(-(\gamma_2 w)^{\beta_2}\right) \end{aligned}$$

where

$$y = \Phi^{-1}\left(1 - \exp\left(-(\gamma_1 w)^{\beta_1}\right)\right)$$

That is, we need to show that, for at least one $w > 0$,

$$\begin{aligned} R \;\; \equiv \;\; & \left(\gamma_1^{\beta_1}/\gamma_2^{\beta_2}\right)(\beta_1/\beta_2)\, w^{\beta_1-\beta_2} \exp\left(-(\gamma_1 w)^{\beta_1}\right) \exp\left((\gamma_2 w)^{\beta_2}\right) \\ & \times \left(\Phi\left((c_{\mathrm{u}} - \mu)/\left(\sigma\sqrt{1-\rho^2}\right) - \rho y/\sqrt{1-\rho^2}\right) - \Phi\left((c_{\mathrm{l}} - \mu)/\left(\sigma\sqrt{1-\rho^2}\right) - \rho y/\sqrt{1-\rho^2}\right)\right) \\ & \div \left(\Phi((c_{\mathrm{u}} - \mu)/\sigma) - \Phi((c_{\mathrm{l}} - \mu)/\sigma)\right) \\ \neq \;\; & 1 \end{aligned} \tag{348}$$

First, consider the case in which $\beta_1 \geq \beta_2$. In this case, as $w \to 0$, $w^{\beta_1-\beta_2}$, $\exp\left(-(\gamma_1 w)^{\beta_1}\right)$, and $\exp\left((\gamma_2 w)^{\beta_2}\right)$ stay bounded while

$$\Phi\left((c_{\mathrm{u}} - \mu)/\left(\sigma\sqrt{1-\rho^2}\right) - \rho y/\sqrt{1-\rho^2}\right) - \Phi\left((c_{\mathrm{l}} - \mu)/\left(\sigma\sqrt{1-\rho^2}\right) - \rho y/\sqrt{1-\rho^2}\right)$$

converges to 0. (We are assuming that $\rho \neq 0$.) Thus, result (348) holds.

Now consider the case in which $\beta_1 < \beta_2$. Let $w \to \infty$. We have

$$\exp\left(-(\gamma_1 w)^{\beta_1}\right) \exp\left((\gamma_2 w)^{\beta_2}\right) = \exp\left(w^{\beta_2}\left(\gamma_2^{\beta_2} - \gamma_1^{\beta_1}/w^{\beta_2-\beta_1}\right)\right)$$

and it is clear that for w large enough, this is greater than

$$\exp\left(k w^{\beta_2}\right) \tag{349}$$

for some $k > 0$.

Next, let $a_u \equiv (c_u - \mu)/\left(\sigma\sqrt{1-\rho^2}\right)$, $a_l \equiv (c_l - \mu)/\left(\sigma\sqrt{1-\rho^2}\right)$, and $b \equiv \rho/\sqrt{1-\rho^2}$. Then, for w large enough,

$$\begin{aligned} S \;\; \equiv \;\; & \Phi\left((c_{\mathrm{u}} - \mu)/\left(\sigma\sqrt{1-\rho^2}\right) - \rho y/\sqrt{1-\rho^2}\right) - \Phi\left((c_{\mathrm{l}} - \mu)/\left(\sigma\sqrt{1-\rho^2}\right) - \rho y/\sqrt{1-\rho^2}\right) \\ = \;\; & \int_{a_l - by}^{a_u - by} \phi(s)\, ds \\ > \;\; & (a_u - a_l) \times \phi(a_l - by) \\ > \;\; & (a_u - a_l) \times \phi(2by) \\ = \;\; & (a_u - a_l) \times \left(1/\sqrt{2\pi}\right)\left(\exp\left(-y^2/2\right)\right)^{4b^2} \end{aligned} \tag{350}$$

Now it is clear from results (348)–(350), that to determine the behavior of R for large w we need only consider

$$T \equiv w^{\beta_1-\beta_2} \exp\left(k w^{\beta_2}\right) (\phi(y))^{4b^2} \tag{351}$$

From Lemma 1, we know that for large w (and thus for large y) we can replace $\phi(y)$ by $(1 - \Phi(y))y$ to obtain

$$\begin{aligned} T \;\; = \;\; & w^{\beta_1-\beta_2} \exp\left(k w^{\beta_2}\right) \left((1 - \Phi(y))y\right)^{4b^2} M(w) \\ = \;\; & w^{\beta_1-\beta_2} \exp\left(k w^{\beta_2}\right) \left(\exp\left(-(\gamma_1 w)^{\beta_1}\right) y\right)^{4b^2} M(w) \\ = \;\; & w^{\beta_1-\beta_2} \exp\left(w^{\beta_2}\left(k - 4b^2 \gamma_1^{\beta_1} w^{\beta_1-\beta_2}\right)\right) y^{4b^2} M(w) \end{aligned} \tag{352}$$

where $M(w) \to 1$ as $w \to \infty$. For large enough w, this is greater than

$$w^{-\beta_2} \exp\left(w^{\beta_2} k/2\right) y^{4b^2} M(w)$$

which (recall that $\beta_2 > 1$) clearly converges to ∞ as $w \to \infty$. Thus, result (348) holds.

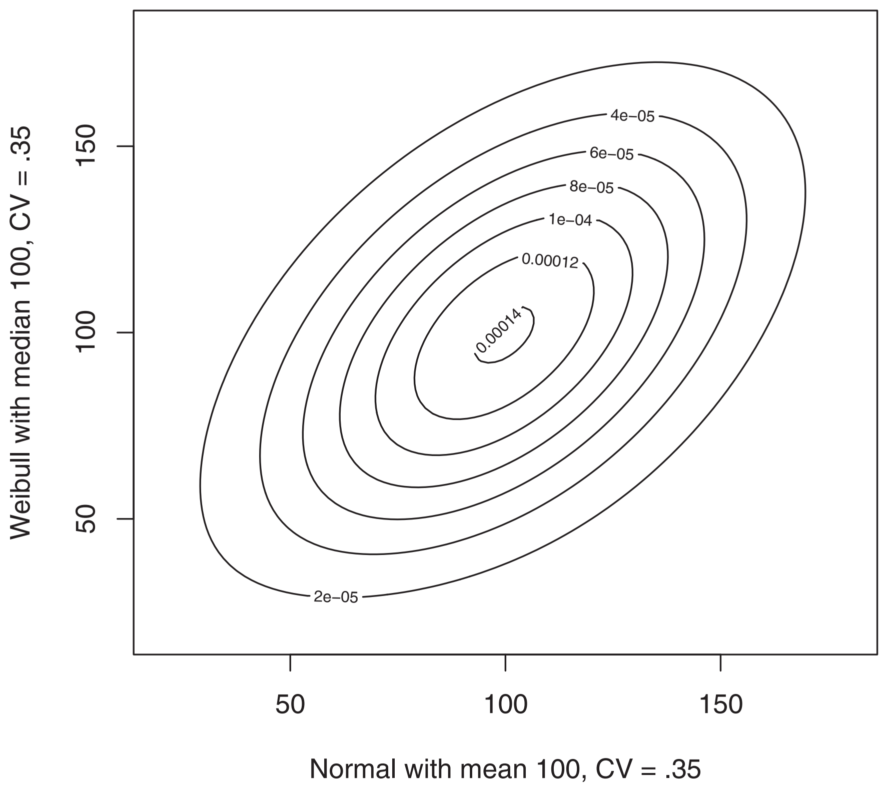

Figure 1: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.35 and a generating correlation of 0.5.

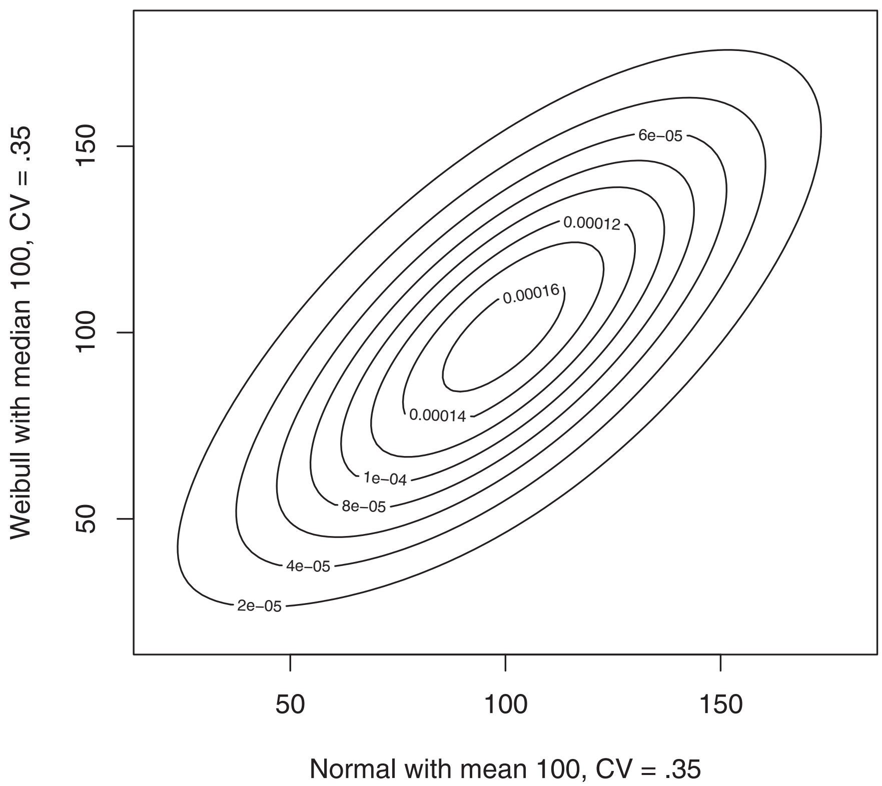

Figure 2: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.35 and a generating correlation of 0.7.

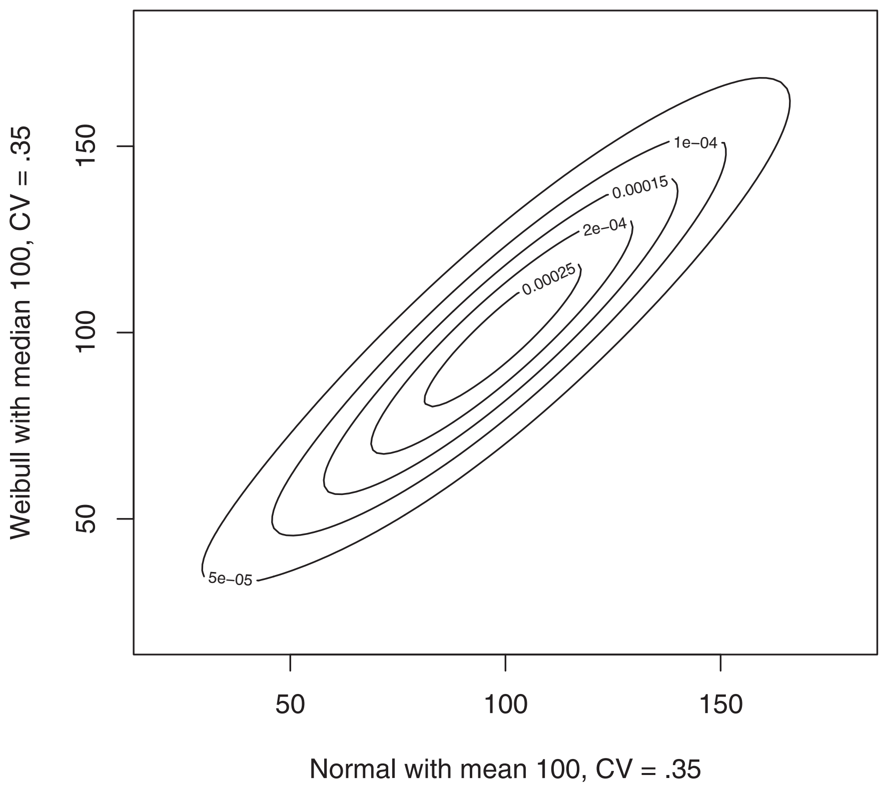

Figure 3: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.35 and a generating correlation of 0.9.

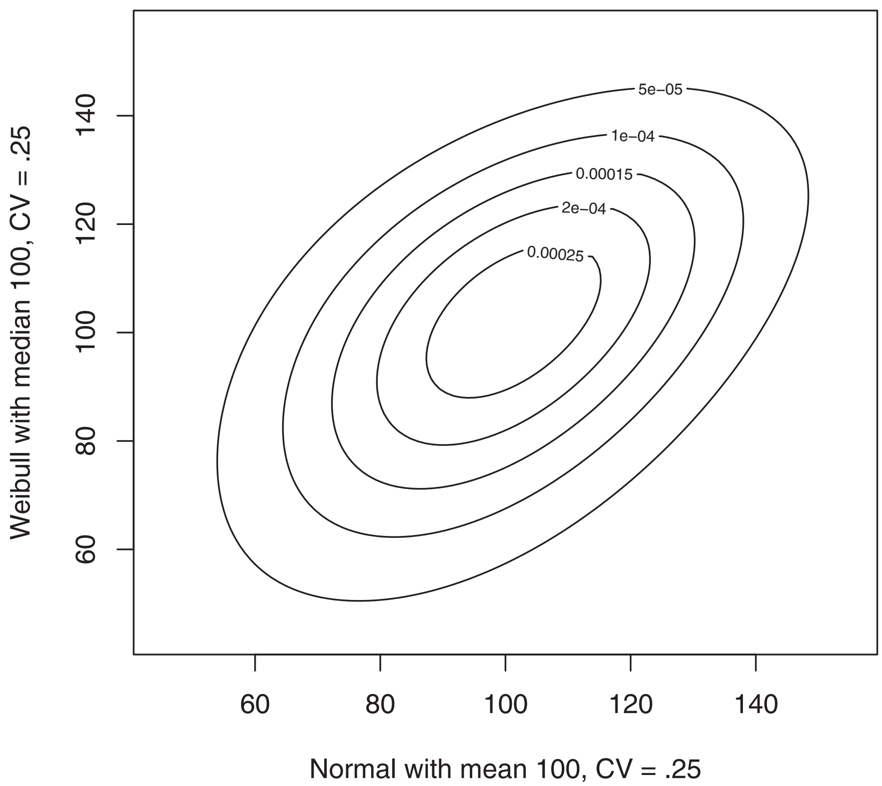

Figure 4: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.25 and a generating correlation of 0.5.

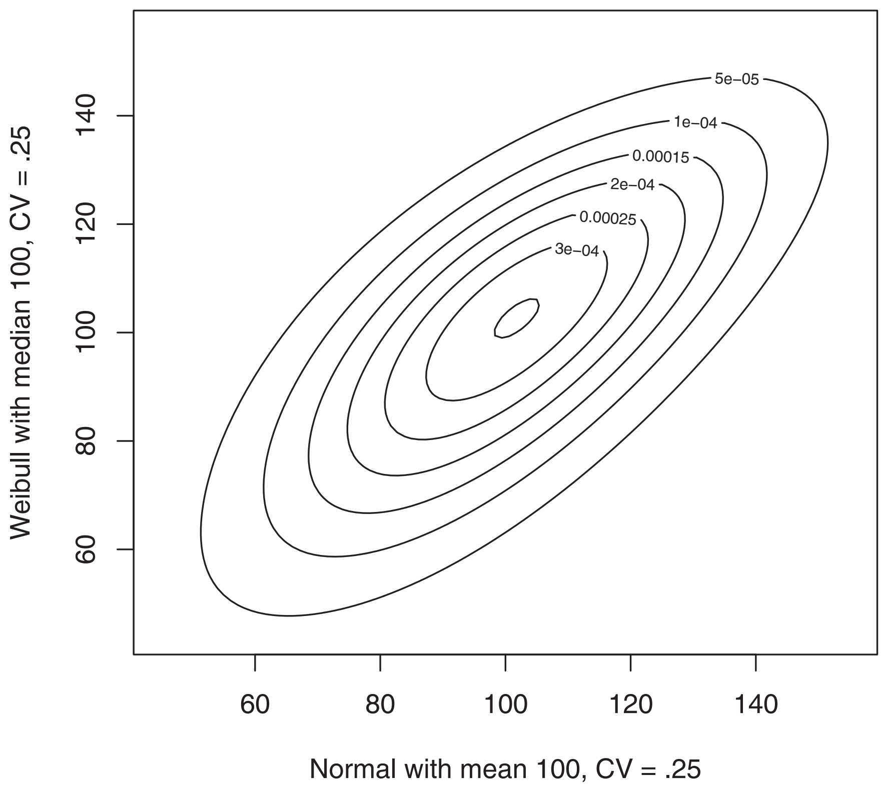

Figure 5: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.25 and a generating correlation of 0.7.

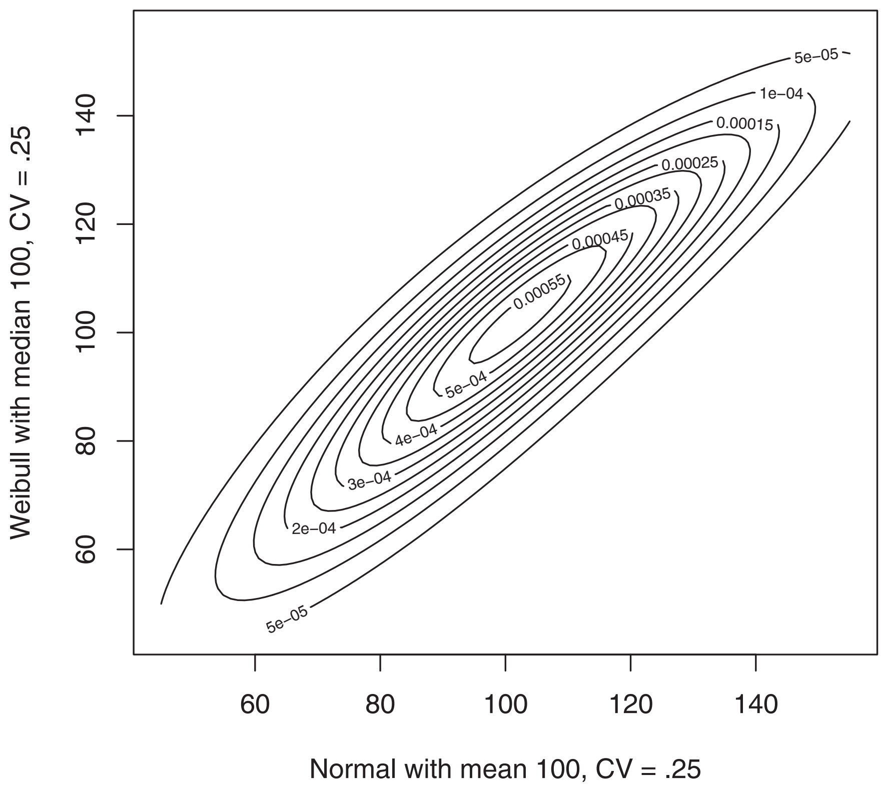

Figure 6: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.25 and a generating correlation of 0.9.

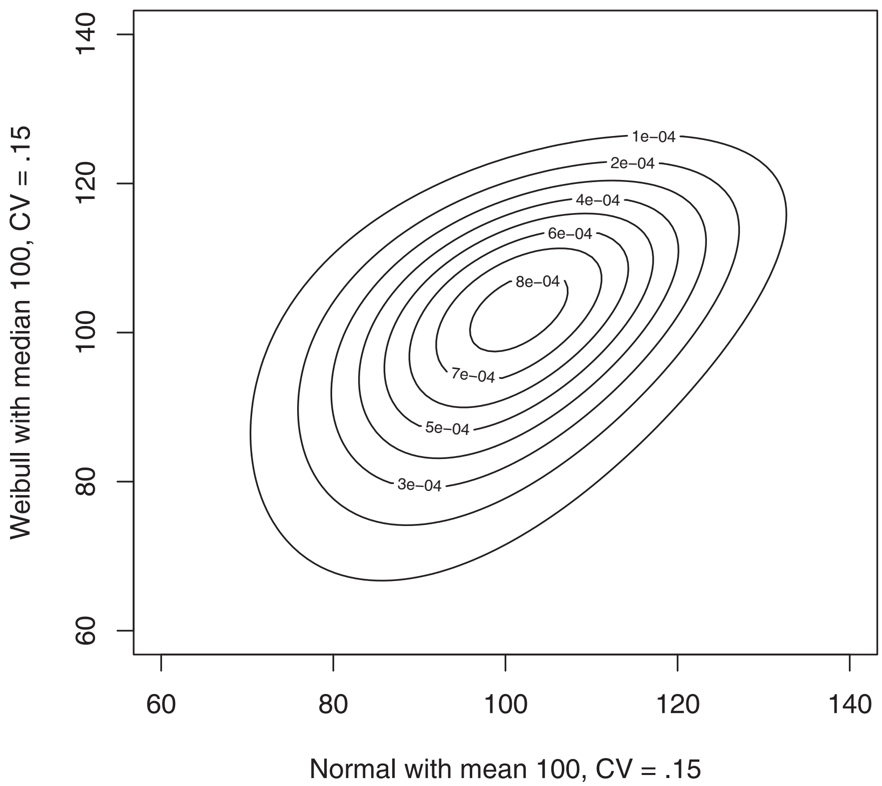

Figure 7: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.15 and a generating correlation of 0.5.

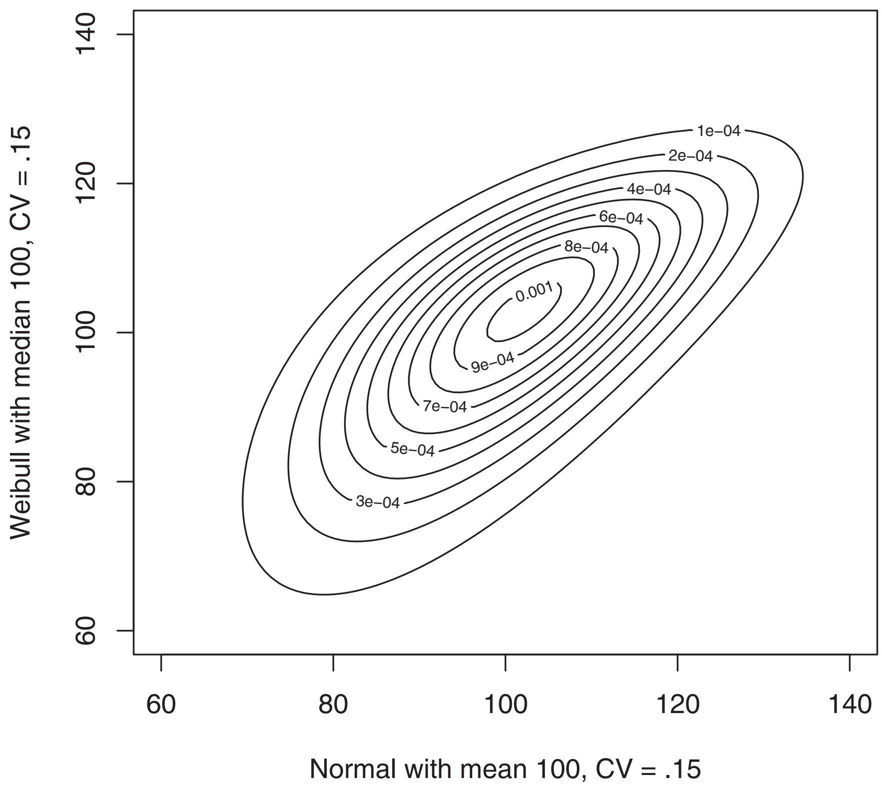

Figure 8: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.15 and a generating correlation of 0.7.

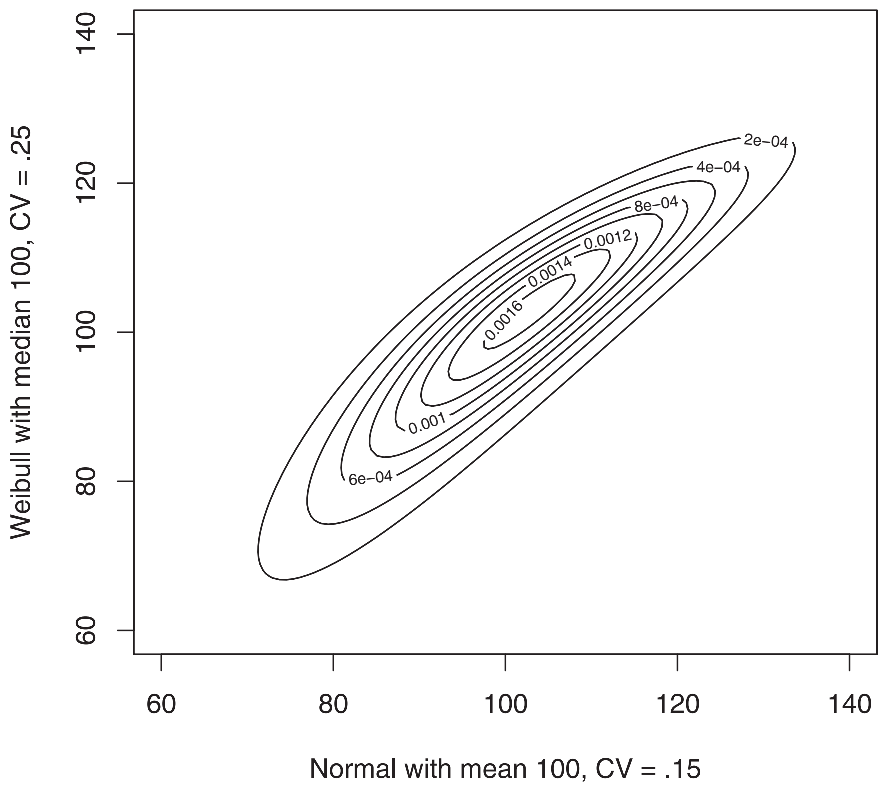

Figure 9: Contour plot of the bivariate Gaussian–Weibull density for Gaussian and Weibull coefficients of variation equal to 0.15 and a generating correlation of 0.9.

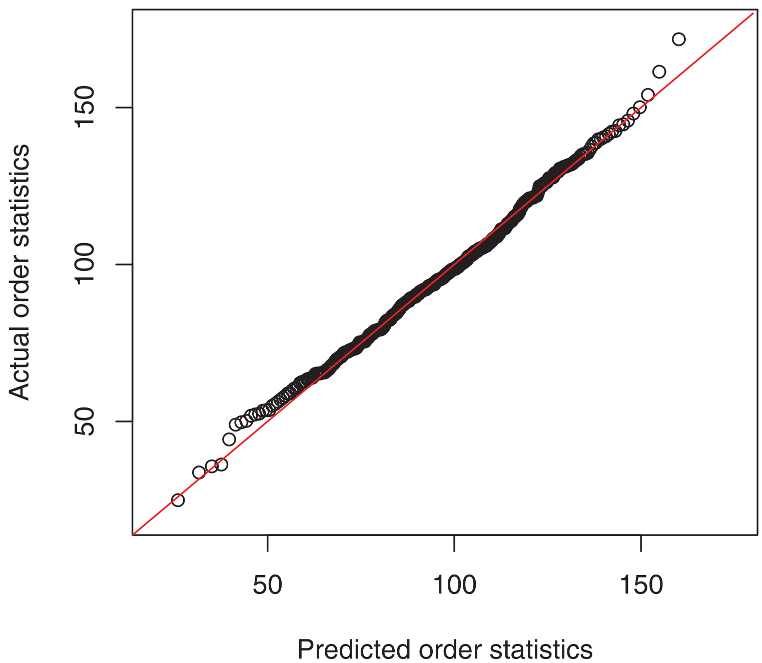

Figure 10: Weibull probability plot of a pseudo-truncated Weibull with generating coefficient of variation equal to 0.25 and generating correlation equal to 0.0. The straight line is the ordinate equals abscissa line.

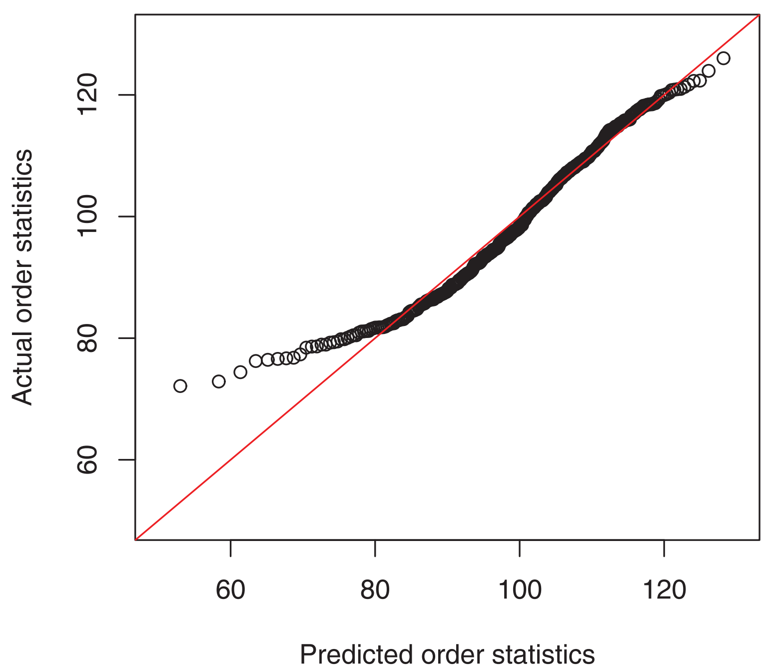

Figure 11: Weibull probability plot of a pseudo-truncated Weibull with generating coefficient of variation equal to 0.25 and generating correlation equal to 0.99. The straight line is the ordinate equals abscissa line.