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SIMPLIFIED METHOD FOR CALCULATING SHEAR DEFLECTIONS OF BEAMS

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Abstract

When one designs with wood, shear deflections can become substantial compared to deflections due to moments, because the modulus of elasticity in bending differs from that in shear by a large amount.

This report presents a simplified energy method to calculate shear deflections in bending members. This simplified approach should help designers decide whether or not shear deflection is important in any complex situation.

Introduction

The purpose of this report is to present a simplified energy method of computing shear deflection in a bending member. Shear deflections due to changing moments often need to be determined for wood, since its modulus of elasticity in bending and in shear differ approximately by a factor of 16. The factor is only 2.5 for steel, and 2.3 to 2.7 for concrete.

A brief review of the theory is presented here to help one to understand the limitations of the method. For more details on energy methods used to compute deflections, see reference (5).[1]

Shear Deflection of Beams

Hooke's law and the well-known shear stress formula from elementary strength of material textbooks give us the total strain energy in a beam of length L due to shear as:

$$U = \frac{1}{2} \int_0^L \int_A \frac{V^2 S^2}{I^2 b^2 G} \, dA dx \qquad (1)$$

where V is the shearing force in the direction of one of the principal axes at any section along the beam due to any general loading through the shear center at that section.

S is the static moment of the cross sectional area, above the point where the shear stress is desired, about the principal axis which is perpendicular to the direction of V.

I is the moment of inertia about to the direction of V.

b is the breadth, at the point where the shearing stress is desired.

A is the cross sectional area of a bean

G is the modulus of elasticity in shear.

Let

$$\int_A \frac{S^2 A}{I^2 b^2} \, dA = k \qquad (2)$$

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[1] Underscored numbers in parentheses refer to Literature Cited at the end of this report.

$\underline{k}$ is constant for a given shape, 10/9 for a circle and 6/5 for a rectangle. (For further detail see Appendix.) Equation (1.) becomes:

$$U = \frac{k}{2} \int_0^L \frac{V^2}{GA}\, dx \tag{3}$$

By applying Castigliano's theorem (5), the deflection Δ_q at point $\underline{q}$ due to a general loading $\underline{P}$ is

$$\Delta_q = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} = k \int_0^L \left.\frac{V}{GA}\right|_{Q=0} \frac{\partial V}{\partial Q}\, dx = k \int_0^L \frac{Vvdx}{GA} \tag{4}$$

where $\underline{V}$ is due to $\underline{P}$ and $\underline{Q}$. Note that $\underline{Q}$ is a fictitious load applied in the direction of and at the point of the desired deflection, and let it be equal to zero after the partial differentiation is performed. In (4), $V\big|_{Q=0}$ is the shear force due to the external loads, call it $\underline{V}$, and $\frac{\partial V}{\partial Q}$ is equivalent to the shear force $\underline{v}$ due to a unit load applied at the point $\underline{q}$ of the desired deflection.

Since $\underline{v}$ is constant for the various segments $\underline{a_i}$ of a beam and since thet quantity $T_i = \int_0^{a_i} \frac{V_i}{G_i A_i}\, dx$ is the area of the shear diagram for the i-th segment due to the general loading $\underline{P}$ divided by $\underline{G_i A_i}$ the shear deflection is:

$$\Delta_q = k \sum_{i=1}^{n} T_i v_i \tag{5}$$

where $\underline{n}$ is the number of segments into which the beam is divided. The magnitude of $\underline{n}$ is determined by any change in $\underline{v_i}$ on one hand or by an abrupt change in $\underline{A_i}$ which effects $\underline{T_i}$.

When a general load has components in the direction of both principal axes, $\underline{k}$ has to be determined with respect to both axes so that the shear deflection can be computed in both directions.

Following are some illustrations of how equation (5) is used if a handbook containing suitable shear diagrams is available.

Example 1. --Statically indeterminate structures. When shear deflections are large relative to the deflections due to moments, the reaction forces found in handbooks are only approximate (6) because they are determined only on the basis of the deflection due to bending moment without considering the shear deflection as is customarily done in textbooks on indeterminate structures.

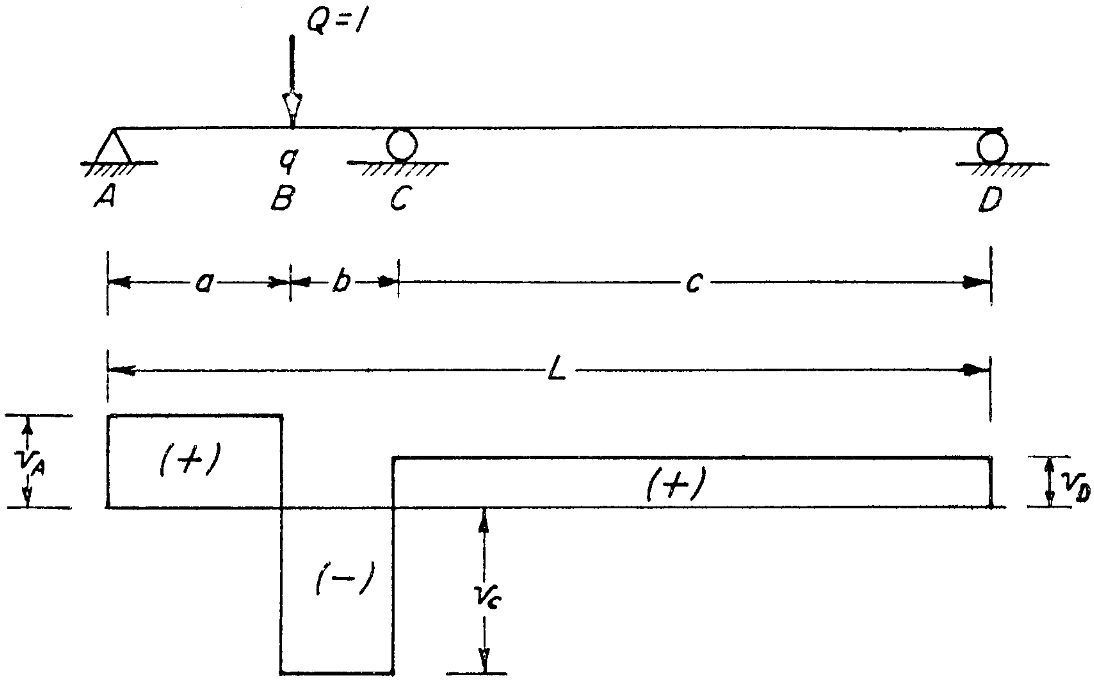

Figure 1.--A statically indeterminate beam and its shear diagram due to unit load at point q.

(M 138 160)

Given A beam supported at three different points, as shown in figure 1, and loaded by any general loading system

Find: The shear deflection at point q.

The approximate magnitude of v_A, v_C, and v_D (as shown in fig. 1) can be found in many handbooks, and their values are dependent on a , b , and c only.

The deflection at q is:

$$\Delta q = k \sum_{i=1}^{n} T_i v_i = k \left[T_{\overline{AB}}\, v_A + T_{\overline{BC}}\, v_C + T_{\overline{CD}}\, v_D \right]$$

Note that the same sign convention has to be used for T_i as for v_i.

Example 2.--Statically determinate structures.

Given: A cantilevered beam, circular in cross section, loaded with a uniformly increasing load, as shown in figure 2.

Find: The deflection due to shear at a point one-third of the span away from the support.

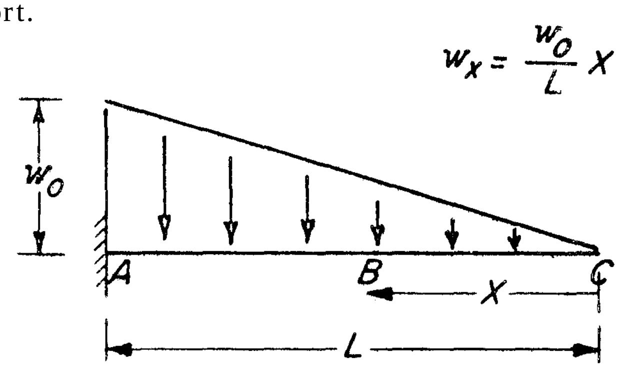

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Figure 2.--Cantilevered beam loaded with a uniformly increasing load.

Figure 3 gives the shear diagram for a unit load at the point of interest.

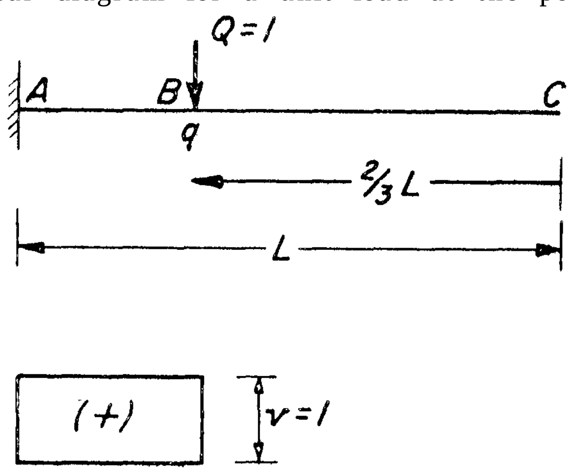

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Figure 3.--Shear diagram due to a unit load at the point of the desired deflection.

Figure 4 shows the shear diagram due to the uniformly increasing load, and the equation for the shear is

$$V = \frac{w_o x^2}{2L}$$

The area of the shear diagram between points A and B (shown by the cross-hatched area in fig. 4) can be determined by the difference between two areas (ABCFD and BCF), remembering that the area under this parabola is one-third of the product of the height and length.

$$T_i = T_{\overline{AB}} = \frac{1}{GA}\left[\frac{w_o L^2}{2L} L \frac{1}{3} - \frac{w_o}{2L}\left(\frac{2}{3} L\right)^2 \frac{2}{3} L \frac{1}{3}\right] = \frac{19 w_o L^2}{162 GA}$$

and since the magnitude of v is 1, the deflection is

$$\Delta_q = kT_1 v_1 = \frac{10}{9}\,\frac{19}{162}\,\frac{w_o L^2}{GA} = \frac{95}{729}\,\frac{w_o L^2}{GA}$$

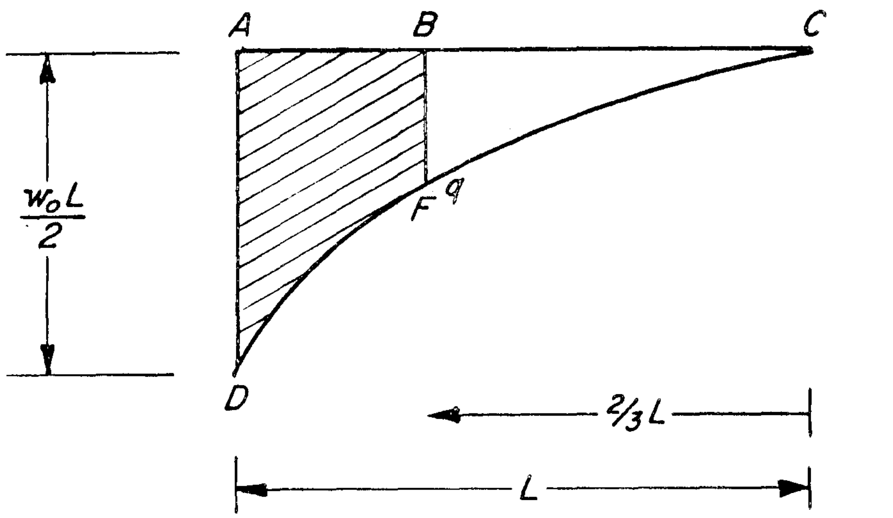

Figure 4.--Shear diagram for uniformly increasing loading.
(M 138 156)

Conclusion

The simplification of the conventional energy method lies in the fact that the integration is reduced to multiplication of the area of the shear diagram (due to a general loading) and the ordinate of the shear diagram due to a unit load applied at the desired point of shear deflection.

The simplification is possible because the ordinate of the shear diagram due to a unit load is constant for various segments of the beam. It is hoped that, with this simplified approach, designers can more readily check shear deflections in cases where computations are ordinarily difficult to make but may be highly desirable, e.g., two-span continuous I-beams made up of plywood web and dimension lumber flanges.

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APPENDIX

Derivation of k for a Circular Cross Section

k can be derived by substituting the static moment, which is expressed as a function of y, and breadth at y, the area of the cross section, and the moment of inertia into equation (2). See figure 1A.

$$k = \frac{A}{I^2}\int_{-r}^{r}\frac{S^2}{b}\,dy = \frac{16}{r^6\pi}\int_{-r}^{r}\frac{\left[\frac{2}{3}\left(r^2 - y^2\right)^{3/2}\right]}{2\sqrt{r^2 - y^2}}\,dy = \frac{10}{9} \tag{1A}$$

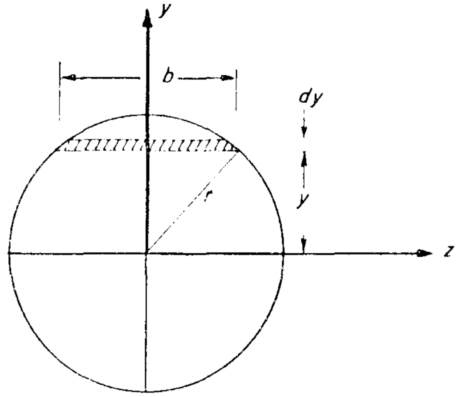

Figure 1A.--Circular cross section showing a differential area and the limits of the integration. (M 138 155)

Method of Determine $\underline{k}$ for an I-Beam

This derivation is applicable when the modulus of elasticity in bending is different for the web and for the flange. In that case, figure 2A refers to the transformed cross section. For further details on transformed cross sections see reference (7). Furthermore, if the modulus of rigidity G_w of the web is also different from the modulus of rigidity G_f of the flange, i.e., if $G_f = \beta G_w$, then the integration has to be done in parts and the limits should be divided up so that the

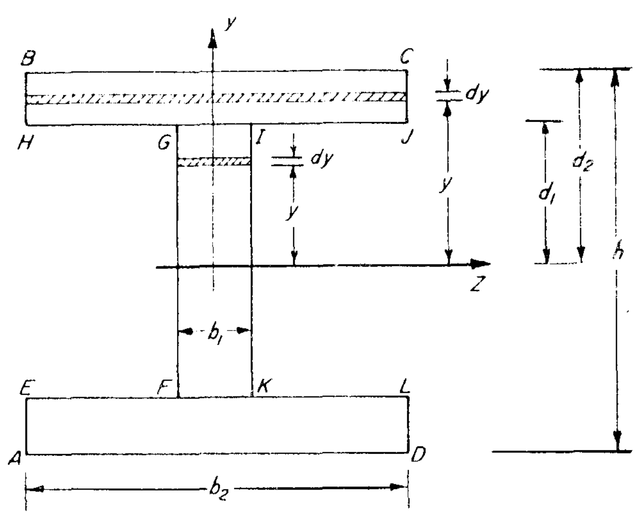

Figure 2A.--Cross section of an I-beam showing differential areas and the limits of the integration for the flange and for the web. (M 138 159)

breadth and G will be constant between any pair of limits. By considering that the I-beam has two axes of symmetry, the k is determined about one of the axes as follows:

$$k = k_1 + k_2 = \frac{2A}{I^2} \int_{d_1}^{d_2} \frac{S_f^2}{b_2 \beta} dy + \frac{2A}{I^2} \int_{o}^{d_1} \frac{S_w^2}{b_1} dy \tag{2A}$$

Note that equation (2A) assumes that $G = G_w$ in equation (4). It is convenient to incorporate β in k According to (2A), S has to be determined in the flange and in the web, where

$$S_f = \int_y^{d_2} y b_2 dy = \frac{b_2}{2}\left(d_2^2 - y^2\right), \; y \geq d_1 \tag{3A}$$

and

$$S_w = \int_{d_1}^{d_2} y b_2 dy + \int_y^{d_1} y b_1 dy$$

$$S_w = \frac{b_2}{2}\left(d_2^2 - d_1^2\right) + \frac{b_1}{2}\left(d_1^2 - y^2\right), \; 0 \leq y \leq d_1 \tag{4A}$$

When k_1 is to be determined, the limits are extended for the height of the flange; therefore, the static moment of only the flange is to be included.

$$k_1 = \frac{2A}{I^2} \int_{d_1}^{d_2} \frac{S_f^2}{\beta b_2} dy = \frac{A}{I^2} \frac{b_2}{30\beta}\left(8d_2^5 - 3d_1^5 + 10 d_2^2 d_1^3 - 15 d_1 d_2^4\right) \tag{5A}$$

For k_2, the static moment in the web should be substituted.

$$k_2 = \frac{2A}{I^2} \int_o^{d_1} \frac{S_w^2}{b_1} dy_1 \tag{6A}$$

that is

$$k_2 = \frac{A}{I^2}\left[\frac{b_2^2}{2b_1}\left(d_2^4 d_1 - 2d_1^3 d_2^2 + d_1^5\right) + b_2\left(d_1^3 d_2^2 - \frac{2}{3}d_1^5 - \frac{d_1^3 d_2^2}{3}\right)\right.$$

$$\left. + \frac{b_1}{2}\left(d_1^5 - \frac{2d_1^5}{3} + \frac{d_1^5}{5}\right)\right] \qquad (7A)$$

If we let $\frac{b_1}{b_2} = p$ and $\frac{d_1}{d_2} = \frac{2d_1}{h} = t$

$$k = \frac{Ab_1h^5}{I^2 32}\left\{\frac{1}{p}\left[\frac{8}{30\beta} - t^5\left(\frac{3}{30\beta} + \frac{20}{30}\right) + t^3\left(\frac{1}{3\beta} + \frac{2}{3}\right) - \frac{t}{2\beta}\right]\right.$$

$$\left. + \frac{1}{2p^2}\left[t - 2t^3 + t^5\right] + \frac{4}{15}t^5\right\} \qquad (8A)$$

The area or the moment of inertia of the I-beam can be obtained as a difference between rectangular ABCD and EFGII plus IJKL, as follows;

$$\frac{A}{I^2} = \frac{hb_2 - 2(b_2 - b_1)d_1}{\left[\frac{b_2h^3}{12} - \frac{(b_2 - b_1)(2d_1)^3}{12}\right]^2} \qquad (9A)$$

By substituting p and t into (9A), and (9A) into (8A) we obtain the final expression for k

$$k = \frac{\frac{9}{2}\left\{\frac{1}{p}(1-t) + t\right\}\left\{\frac{1}{p^2}\left[\frac{t^5}{2} - t^3 + \frac{t}{2}\right] + \frac{1}{p}\left[-t^5\left(\frac{3}{30\beta} + \frac{2}{3}\right) + t^3\left(\frac{1}{3\beta} + \frac{2}{3}\right) - \frac{t}{2\beta} + \frac{8}{30\beta}\right] + \frac{8t^5}{30}\right\}}{\left[\frac{1}{p}(1 - t^3) + t^3\right]^2} \qquad (10A)$$

which reduces to $\frac{6}{5}$ if $\beta = 1$, $t = 1$, and $p = 1$, or simply a rectangular cross section. Note that k is independent of the actual sizes of the I-beam and depends only on the ratio of the dimensions and on the ratio of the G's. Equation (10A) is presented in graphic form, in figure 3A.

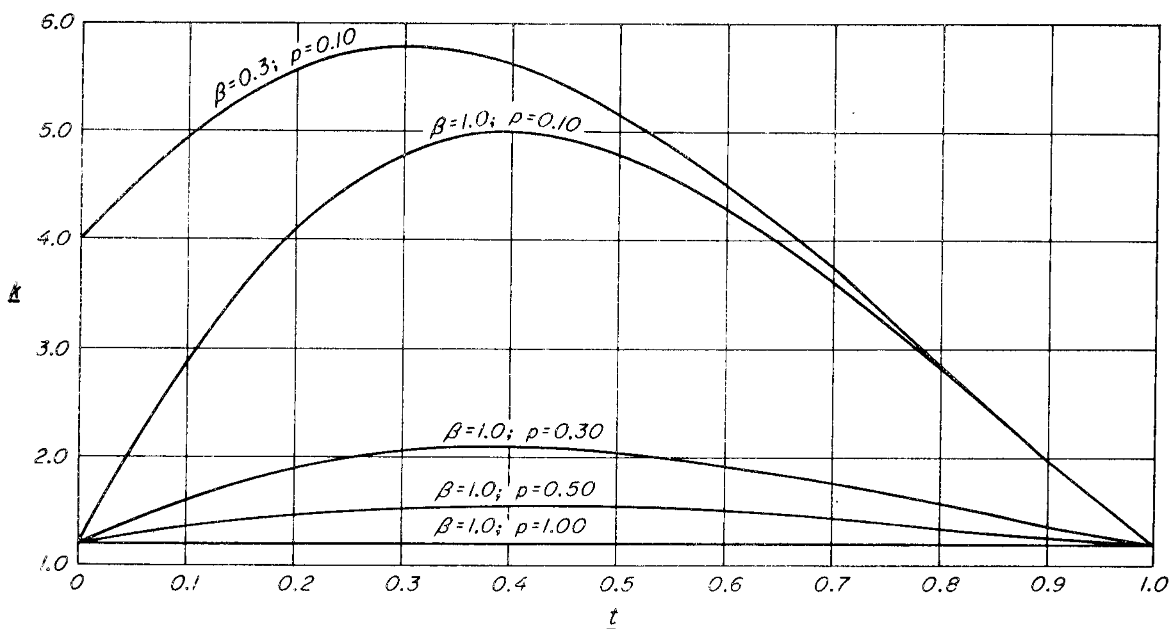

Figure 3A.--Graphic presentation of equation (10A).

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