

FORMULAS FOR COLUMNS WITH SIDE LOADS AND ECCENTRICITY

November 1950



No. R1782

UNITED STATES DEPARTMENT OF AGRICULTURE
FOREST SERVICE
FOREST PRODUCTS LABORATORY
Madison 5, Wisconsin

In Cooperation with the University of Wisconsin

FORMULAS FOR COLUMNS WITH SIDE LOADS AND ECCENTRICITY¹

By

LYMAN W. WOOD, Engineer

Forest Products Laboratory, ² Forest Service
U. S. Department of Agriculture

Introduction

The design of columns with side loads and eccentric end loads is an old problem that has received attention from many authors. General textbooks such as those by Church³ or Maurer and Withey⁴ and the more specialized work of Salmon on columns are among the many publications that have dealt with the subject. The treatment of short columns, considered as prisms and with stresses due to deflection neglected, is relatively simple, but long columns with deflection stresses are more complex.

Critical loads on centrally loaded long columns that fail, by bending can be calculated satisfactorily with the Euler formula. Additional formulas of satisfactory accuracy have been developed for long columns with eccentric end load or with certain specific combinations of side load with eccentric end load.² These latter formulas, however, require rather cumbersome trial calculations involving the secant of an angle that can be determined only indirectly; they have the further disadvantage that a separate formula must be developed for each condition of side loading.

Recognizing the difficulty in applying the secant formulas to problems of safe loads, on long columns with side loads and eccentricity, Newlin¹ sought a sim-

¹ This is an amplification and explanation of an article of the same title by J. A. Newlin, formerly Chief, Division of Timber Mechanics, Forest Products Laboratory, published in Building Standards Monthly, December 1940. The formulas are also given in "National Design Specification for Stress-grade Lumber and Its Fastenings," published by the National Lumber Manufacturers Association. Acknowledgment is made to C. B. Norris of the Forest Products Laboratory staff for review and confirmation of Newlin's analysis. First report of this number published in 1950.

² Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

³ Church, Irving P., Mechanics of Engineering, New York, 1914.

⁴ Maurer, Edward R. and Withey, Morton O., Strength of Materials, New York, 1940.

⁵ Salmon, E. H., Columns, London England, 1921.

lified method of design. He succeeded in developing a general formula in which eocentrioity is simply represented and in which any condition of side loading can be expressed in terms of the bending stress induced by it. His formula, while somewhat more approximate than the secant formulas, has as much accuracy as exists in the present state of knowledge of the properties of wood or of the conditions under which wood columns are loaded. The formula is not difficult to apply and is believed to be highly useful in the design of wood columns.

Newlin's original report and published article were rather brief, and a number of steps in his derivation of the long-column formula were omitted. The derivation of a similar formula for short columns was also omitted. The present report has been prepared to fill in the omitted material and to illustrate the application of the formulas by means of suitable examples.

Notation

(All units except ratios are in pounds, inches, or combinations thereof)

A = area of the cross section of a column.

C = unit failing stress in compression parallel to grain.

E = Young's modulus of elasticity of the column material.

F = unit failing stress flexure.

I = moment of inertia of the cross section of a column around the neutral axis. As used here, the neutral axis is perpendicular to the direction of eccentricity or side load. In a rectangular column,

$$I = \frac{bd^3}{12} .$$

K = value of the slenderness ratio $\frac{L}{d}$ dividing intermediate from long columns.

L = length of a column or, more specifically, unsupported length.

$\frac{L}{d}$ = slenderness ratio of a column. As used here, d is measured in the direction of eccentricity or side loads.

M = generally, the bending moment on a column from eccentricity or side load, but is used here also in the more restricted sense of bending moment caused by the portion of side load that is independent of axial load.

$\frac{M}{S}$ = generally, the flexural stress induced in the outer fiber of a column from eccentricity or side load, but used here also in the more restricted sense of stress in the outer fiber induced by the portion of side load that is independent of axial load.

P = axial load (end load) on a column, whether centrally or eccentrically applied.

$\frac{P}{A}$ = direct compressive stress induced by axial load (end Load) on a column.

S = section modulus of the cross section of a column related to flexure in the direction of eccentricity or side loads. Section modulus of a rectangular column is $\frac{bd^2}{6}$.

W = critical Euler load on a pin-ended column. $W = \frac{\pi^2 EI}{L^2}$

b = width of a rectangular column.

c = allowable unit working stress in compression parallel to grain for a column of the length under consideration with centrally applied axial load and no side load; this may be a short-column, intermediate-column, or long-column stress.

d = depth of a rectangular column) measured in the direction of eccentricity or side load.

d = additional deflection at midlength of a long column due to axial load.

e = eccentricity, the distance from the center of gravity of the column section to the center of gravity of the applied load. Eccentricity in this analysis is considered as being parallel to one of the sides of a rectangular column.

$\epsilon_1, \epsilon_2, \epsilon_3$ = increments of deflection at midlength of a column in Salmon's analysis.

f = allowable unit working stress in flexure, as in a simple beam with transverse loading only.

f_0 = flexural stress corresponding to the total deflection at midlength of a long column when all eccentric and side loads are on.

f_1, f_2, f_3 = increments of flexural stress corresponding to increments of deflection $\epsilon_1, \epsilon_2,$ and ϵ_3 .

f_e = fictitious flexural stress due to eccentricity of axial load assumed to replace the original deflection of the column in Salmon's analysis.

m, n = exponents in the general form of the equation for strength under combined loading.

w = critical Euler unit stress on a pin-ended column. $w = \frac{W}{A}$.

y_0 = total deflection at midlength of a column in Salmon's analysis when all loads are on.

z = ratio of flexural to direct compressive stress; that is, $\frac{M/S}{P/A}$ more specifically, a ratio of flexural stress to direct compressive stress when both result from the same loading, so that the ratio remains constant while the load varies.

Analysis of Combined Stresses

Textbooks on strength of materials ⁴ show how the combined stress on a prism or short column due to axial and flexural loads is represented by the sum $\frac{M/S}{P/A}$. In this expression, M is bending moment, which may be induced by direct-acting side loads, by eccentricity of the axial load, or by both. For purposes of design in some structural materials, the sum $\frac{M/S}{P/A}$ is not permitted to exceed a safe stress value, assumed to be the same in bending as in compression.

Where strength in bending is unequal to that in compression, as is the case with wood, there is a problem of determining what is the ultimate strength

value for various combinations of load. It is recognized that the strength value under combined stress lies generally somewhere between the two values for separated stresses and is influenced by the relative proportions of each. The relationship which describes the condition for failure under combined stress may be of the interaction type.

$$+ \left(\frac{P/A}{C}\right)^n = 1 \quad (1)$$

where M/S and P/A are stresses applied in flexure and compression and F and C are the corresponding ultimate strength values under separate loading. Since F and C are known values, either M/S or P/A can be calculated by assuming a value for the other. Then the sum $M/S + P/A$ becomes the strength under the assumed combination of load.

Newlin and Trayer⁶ made tests of clear Sitka spruce under combined loading which showed the relation of the strength to the relative proportions of bending and axial stress. The curve of maximum stress from figure 6 of their report,⁶ as obtained from these test results, is shown as a solid line in figure 1.

The most simple equation of type (1) is with exponents m and n each equal to unity; this relation gives combined strength values in clear Sitka spruce indicated by the lower broken-line curve in figure 1. A much closer agreement with the test results is obtained if $m = 2$, giving the upper broken-line curve of figure 1.

From consideration of figure 1, it is apparent that strength under combined stress is estimated very closely from the relationship

$$\left(\frac{M/S}{F}\right)^2 + \frac{P/A}{C} = 1 \quad (2)$$

This relationship, however, is somewhat complex to handle as a formula, especially when M/S is broken up into two or three components representing side loads and eccentricity of axial load. Newlin's formulas: for short columns indicate that he used the form

$$\frac{M/S}{F} + \frac{P/A}{C} = 1 \quad (3)$$

which is simpler of application. This relationship (3) has also had extensive use by other designers dealing with the problem of combined loads. Figure 1 shows that it gives rather conservative estimates of strength, but not far in error, with the lower ratios of bending to total stress such as are most likely to occur in short columns.

⁶Newlin, J. A. and Trayer, G. W., Stresses In Wood Members Subjected to Combined Colume and Beam Action, Forest Products Laboratory Report No. 1311, 1941.

In the case of long columns, Newlin simplified his safe-loading formula by assuming that the ultimate strength under combined load is equal to the bending strength. This is explained as follows. Long columns are those in the range in which the Euler formula applies. The Euler formula is based on elasticity rather than strength and can be applied only if P/A is less than the proportional-limit stress in compression. This stress is usually assumed to be two-thirds of the compressive strength of wood, ¹ or $2C/3$ by the notation of this report. The shortest column in the Euler class will reach its maximum load at this stress. The unit flexural strength, being a property of the material, is unaffected by the length of the column. When stress due to an axial load on a long column equals $2C/3$, and if it is assumed that the maximum stress developed under combined load is equal to the flexural strength, the available bending stress is $F - 2C/3$, and the ratio of bending stress to total stress becomes

$$\frac{F - 2C/3}{F}$$

This is a minimum value for that ratio in long columns. In the case of clear Sitka spruce (fig. 1), the minimum ratio is

$$\frac{7,900 - 2,867}{7,900} = 0.64$$

It follows that the long-column formula leads to ratios within the range of 0.64 to 1.00, as indicated by the heavy horizontal line at the top of figure 1. Corresponding ratios based on safe rather than ultimate stresses and applicable to other species range from about 0.55 to 1.00, as indicated on figure 1. It can be seen that, in the range from 0.55 to 1.00, the maximum stress actually developed in test is not much short of the flexural strength.

If the relation expressed in equation (2) were used to estimate maximum stress in a long column, a formula of considerable complexity would result. Equation (3) would give a usable formula, but would result in overly conservative estimates of load. The assumption that strength under combined loading of long columns equals flexural strength has therefore been chosen.

Short Columns

Wood columns with slenderness ratios (ratio of unsupported length to least dimension of cross section) of 11 or less receive both compression and flexural stresses under eccentric axial load, but it is not necessary to assume any additional stress due to deflection. The combined stress, derived as for prisms by well-known principles of mechanics, ⁴ is expressed by the quantity $P/A + M/S$. In this expression M is bending moment, which may be induced by eccentricity of an axial load, by direct-acting side loads, or by both. For maximum safe load, the relationship of equation (3) is used with safe stresses instead of ultimate strength values.

¹ Newlin, J. A. and Gahagan, J. M., Tests of Large Timber Columns and Presentation of the Forest Products Laboratory Column Formula. U. S. Dept. of Agr. Tech. Bull. No. 167, 1930.

$$\frac{M/S + P/A}{f} = 1 \quad (4)$$

Equation (4) is directly applicable where there are side loads and concentric axial load. If the end load is proportional to the side load (as in upper chord members of roof trusses, with bending stress induced by roof loads acting through purlins, and with axial loads proportional to the same roof loads), M/S can be replaced by $z P/A$, and equation (4) becomes

$$\frac{z (P/A) + P/A}{f} = 1 \quad (5)$$

The bending stress M/S may be induced by eccentricity of the axial load instead of by side load. In that case, the external moment M is expressed by the quantity, Pe . The section modulus S may be replaced by $\frac{I}{d/2}$ (d measured in the direction of eccentricity). Then

$$M/S = \frac{Pe d}{2 I} = \frac{P}{A} \left(\frac{A e d}{2 I} \right) \quad (6)$$

But in a rectangular column, $A = bd$ and $I = \frac{bd^3}{12}$. Substitution of these values in (6) gives

$$M/S = P/A \left(\frac{6e}{d} \right) \quad (7)$$

Substitution of expression (7) in equation (4) gives

$$\frac{P/A (6 e/d) + P/A}{f} = 1 \quad (8)$$

for the condition of eccentric axial load and no side load.

Where both side load and eccentric axial load are present the term M/S in (4) represents the total of bending stress from both. Furthermore, the side load may be considered as composed of two portions, one independent of axial load (M/S) and the other proportional to axial load ($z P/A$). The total bending stress is thus expressed by

$$P/A \left(\frac{6e}{d} \right) + M/S + z P/A$$

and equation (4) becomes

$$\frac{P/A (6 e/d) + M/S + z P/A}{f} + \frac{P/A}{c} = 1 \quad (9)$$

where M/S has the more restricted meaning of bending stress due to that portion of side load which is independent of axial load.

Equation (9) is a general equation from which any of the others is readily derived by dropping certain terms. For example, if all side load is proportional to axial load, and axial load is eccentrically applied, the term M/S is dropped. If all side load is proportional to axial load, and axial load is centrally

applied, both M/S and $P/A \left(\frac{6e}{d}\right)$ are dropped, and equation (5) results. If there is eccentricity but no side load, M/S and $\frac{P}{A}$ are dropped, and equation (8) results. Other loading conditions can be similarly represented by suitable selection of terms. Where there is neither side load nor eccentricity, all three terms in the numerator of the first fraction of (9) disappear, leaving only

$$\frac{P/A}{c} = 1 \text{ or } \frac{P}{A} = c.$$

It is to be noted that the term $\frac{P}{A} \left(\frac{6e}{d}\right)$ in equation (9) is developed from consideration of a rectangular cross section. Equation (9) is therefore applicable in this form only to columns of square-rectangular cross section.

Any of the foregoing equations can be solved, either for P/A or for M/S , if desired to facilitate any particular design problem. Solution for P/A or M/S will be easier if any redundant terms are first dropped. Newlin published these equations in a form in which they are solved for P/A .¹

Long Columns

In long columns of the Euler class, the problem of eccentric and side loads is made more complex by the addition of stress induced by the curvature of the column itself. The secant formulas² are applicable but rather difficult to use in problems of determination of safe load. Furthermore, it is desirable to express all conditions of side loading in terms of their resulting moments or stresses, thus making one general formula applicable to all loading conditions. Newlin¹ succeeded in doing this.

In developing a general formula for the condition of side and eccentric axial loads, use is made of a relationship developed by Salmon³ for columns bent to sinusoidal curvature. While it is recognized that the most common side loadings do not produce sinusoidal curvature, Salmon showed that with small amounts of curvature, the deflection is practically the same whether the elastic curve is circular, parabolic, or sinusoidal. The error introduced by assuming sinusoidal curvature is probably of lesser magnitude than approximations in the present state of knowledge on the strength properties of wood or on the conditions under which wood columns are loaded.

Salmon considered a column originally bent to a sinusoidal curvature with a deflection ϵ_1 at midlength (fig. 2). Under the action of an axial load P , the de-

deflection increases to a value y_0 . The original curvature of the column is represented by the equation

$$y = \epsilon_1 \cos \frac{\pi}{L} x \quad (10)$$

For this condition, Salmon arrived at the relationship

$$y_0 = \frac{\epsilon_1}{1 - P/W} \quad (11)$$

where P is the axial load on the column and W is the critical Euler load (notation not the same as that used by Salmon), By algebraic transformation, equation (11) becomes

$$\frac{\epsilon_1}{y_0} + \frac{P}{W} = 1 \quad (12)$$

One property of a sinusoidal elastic curve is that deflection is proportional to the stress causing it. Taking, for example, the equation for sinusoidal curve (10), there follows by differentiation

$$\frac{d^2 y}{dx^2} = -\epsilon_1 \frac{\pi^2}{L^2} \cos \frac{\pi}{L} x = -\frac{\pi^2}{L^2} y \quad (13)$$

Since this is an elastic curve, the general equation of external and internal moments in bending may be applied:

$$M = -EI \frac{d^2 y}{dx^2} \quad (14)$$

Substituting $\frac{d^2 y}{dx^2}$ from equation (13)

$$M = EI \frac{\pi^2}{L^2} y \quad (15)$$

From the general expression for bending stress equal to M/S ,

$$\text{stress} = \frac{EI}{S} \frac{\pi^2}{L^2} y \quad (16)$$

Equation (15) shows the proportionality of moment, and equation (16) shows the proportionality of stress to deflection.

Newlin used Salmon's originally bent column (fig. 2) to represent an eccentrically loaded long column with axial load P and an eccentricity corresponding to the deflection ϵ_1 . If side loads are applied to such a column, additional deflections ϵ_2 and ϵ_3 may result. If curvature remains sinusoidal, the summation of deflections $\epsilon_1 + \epsilon_2 + \epsilon_3$ will hold the same relationship to y_o as ϵ_1 holds in equation (12). Then

$$\frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{y_o} + \frac{P}{W} = 1 \quad (17)$$

It has already been shown in equation (16) that, with sinusoidal curvature, deflection is proportional to stress. Since each of the curves in figure 2 is assumed to be sinusoidal, all of the deflections ϵ_1 , ϵ_2 , and ϵ_3 , and y_o have the same factor of proportionality to the bending stresses that cause them the deflection ratio

$$\frac{\epsilon_1 + \epsilon_2 + \epsilon_3}{y_o}$$

is thus equal to a stress ratio

$$\frac{f_1 + f_2 + f_3}{f_o}$$

The load ratio P/W can also be replaced by the corresponding stress ratio $\frac{P/A}{w}$ where w is the critical Euler stress. Equation (17) then becomes

$$\frac{f_1 + f_2 + f_3}{f_o} + \frac{P/A}{w} = 1 \quad (18)$$

Now let f_1 be replaced by a fictitious stress f_e , which, if it had been present in Salmon's column, would have caused the deflection ϵ_1 . The fictitious stress f_e which could have caused deflection ϵ_1 can be evaluated in terms of the critical Euler load by substituting f_e and ϵ_1 in equation (16) for a sinusoidal elastic curve

$$f_e = \frac{EI}{S} \frac{\pi^2}{L^2} \epsilon_1 \quad (19)$$

It is now necessary to determine what eccentricity e on a straight column corresponds to the deflection ϵ_1 in Salmon's originally bent column. The two

conditions are compared in figure 3. Consider Salmon's column (fig. 3A) with an original deflection ϵ_1 and bent under a small load P to an additional deflection δ with sinusoidal curvature. From equation (12)

$$\frac{\epsilon_1}{\delta + \epsilon_1} + \frac{P}{W} = 1 \quad (20)$$

By algebraic transformation

$$\delta = \epsilon_1 \left[\frac{P/W}{1 - P/W} \right] \quad (21)$$

which, when P/W is small, becomes approximately

$$\delta = \epsilon_1 (P/W) \quad (22)$$

If the column were originally straight with the same load P applied with an eccentricity e such that the same deflection δ resulted (fig. 3B), there is obtained from the secant formula for eccentric load²

$$\delta = e \left(\sec \frac{L}{2} \sqrt{\frac{P}{EI}} - 1 \right) = e \left(\sec \sqrt{\frac{PL^2}{4EI}} - 1 \right) \quad (23)$$

Since in the Euler formula

$$W = wA = \frac{\pi^2 EI}{L_2^2} \quad \text{and} \quad \frac{L^2}{4EI} + \frac{\pi^2}{4W} \quad (24)$$

Substituting (24) in (23) gives

$$\delta = e \left(\sec \sqrt{\frac{\pi^2 P}{4W}} - 1 \right) \quad (25)$$

Tables of integrals⁸ express the secant of an angle in terms of the series

$$\sec x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \dots \quad (26)$$

With P/W small, terms beyond the first two of (26) are negligible, and (25) becomes very nearly

$$\delta = \left(1 + \frac{\pi^2 P}{8W} - 1 \right) = e \left(\frac{\pi^2 P}{8W} \right) \quad (27)$$

⁸ Pierce, B. O., A Short Table of Integrals, third edition revised, New York, 1929.

Equating (27) with (22) gives

$$\epsilon_1 \frac{P}{W} = e \left(\frac{\pi^2 P}{8W} \right) \quad (28)$$

Since $\frac{\pi^2}{8}$ is very nearly $\frac{5}{4}$, $\epsilon_1 = \frac{5}{4} e$ (29)

Newlin estimated the relationship of e to ϵ_1 from a different basis,¹ but arrived at the same values as in (29), pointing out that ϵ_1 could be replaced by $\frac{5e}{4}$ with results that would check quite closely with the secant formula.

Substitution of (29) in (19) gives

$$f_e = \frac{5EI \pi^2 e}{4SL^2} \quad (30)$$

Substitution of expression (24) for the critical Euler load on a pin-ended column in (30) gives

$$f_e = wA \left(\frac{5e}{4S} \right) \quad (31)$$

In a rectangular column, $A = bd$ and $S = \frac{bd^2}{6}$ (d measured in the direction of eccentricity of the load), from which

$$f_e = w \left(\frac{15e}{2d} \right) \quad (32)$$

Now let f_2 in equation (18) be replaced by M/S , a stress induced by a side load that is independent of the axial load. Let f_3 be replaced by $z P/A$, a stress induced by a side load that is proportional to the axial load. Note that either M/S or $z P/A$ may be entirely missing in actual loading conditions.

Substituting the equivalents for $f_1 = f_e$, f_2 , and f_3 in equation (18) gives

$$\frac{w \left(\frac{15e}{2d} \right) + M/S + z P/A}{f_o} + \frac{P/A}{W} = 1 \quad (33)$$

Equation (33) expresses the condition for breaking load, since w is the Euler critical or breaking stress. The same relation holds for safe or design load by substituting c , the Euler safe working stress, for w . Equation (33) then becomes

$$\frac{c \left(\frac{15e}{2d} \right) + M/S + z P/A}{f_o} + \frac{P/A}{c} = 1 \quad (34)$$

The available value for the bending stress f_o when all loads are on is now limited by the safe bending stress value f . In the extreme fiber on the concave side of the column) f_o and P/A are additive, and their sum cannot be permitted to exceed f (as previously noted under "Analysis of Combined Stresses") analysis leading to equation (11) f_e was not applied, so for Salmon's relationship to hold, f_e or its equivalent $c (\frac{15e}{2d})$ should be deducted from the sum $f_o + P/A$. Then, for the condition of safe loading, the sum $f_o + P/A - c (\frac{15e}{2d})$ cannot exceed f . To solve for the maximum safe load using the assumption developed under "Analysis of Combined Stresses,"

$$f_o + P/A - c (\frac{15e}{2d}) = f$$

By transposition, $f_o = f - P/A + c (\frac{15e}{2d})$ and equation (34) then becomes

$$\frac{c (\frac{15e}{2d}) + M/S + z P/A}{f - P/A + c (\frac{15e}{2d})} + \frac{P/A}{c} = 1 \quad (35)$$

Algebraic transformation of equation (35) by clearing fractions and collecting terms gives a more simple form

$$\frac{P/A (\frac{15e}{2d}) + M/S + z P/A}{f - P/A} + \frac{P/A}{c} = 1 \quad (36)$$

the general equation for the maximum safe value of any combination of side loads and eccentricity on long columns.

Equation (36) is in a general form from which equations for a number of special cases can be readily derived by dropping the inapplicable terms. For example, if all side load is proportional to axial load, and axial load is eccentrically applied, the term M/S is dropped. If all side load is proportional to axial load, and axial load is centrally applied, both M/S and $P/A (\frac{15e}{2d})$ are dropped.

Where there is eccentricity but no side load, both M/S and $z (P/A)$ are dropped. Other loading conditions can be similarly represented by suitable selection of terms. Where there is neither side load nor eccentricity, all three terms in the numerator of the first fraction of (36) disappear, leaving only

$\frac{P/A}{c} = 1$ or $P/A = c$. These are the same processes as in equation (9) for short columns; it is to be remembered, however, that c in equation (36) is the safe long-column stress, while c in equation (9) is the safe short-column stress.

Because of the method of its derivation equation (36), like equation (9), is applicable only to columns of square or rectangular cross section.

Equation (36) or any of the specialized equations derived from it can be solved for P/A or for M/Sc , if desired to facilitate my particular design problem. It is to be noted that solution for P/A results in most cases in a quadratic equation, for which there are two roots of the form

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The root using the minus sign before the radical is used, since the root using the plus sign would give an impossible result. Newlin published these equations in a form in which they are solved for P/A . ¹

Equation (36) is developed from the assumption that maximum moment and maximum deflection occur at midlength of the column. This is true for eccentric axial load and for side load applied symmetrically along the length, but is not true with large side loads unsymmetrically placed along the length of the column. Recognizing this point, Newlin made the following recommendations in regard to unsymmetrically applied side load.

1. For a single concentrated side load, the stress under the load can be used, regardless of the position of the side load with reference to the length of the column. Flexural stress from side load is maximum at this point.
2. The stress to use with a system of side loads is the maximum stress due to the system. Where the system of side loads is such that maximum moment from side load is present near the end of the column, some slight error on the side of overload will occur.

Columns of Intermediate Length

Newlin chose a lower limiting value of 20 for the slenderness ratio $(\frac{L}{d})$ in his long-column formula (36). This limit was chosen as being approximately the lower limit for values of K (slenderness ratio separating intermediate- and long-column groups) in the most common species and grades for structural use. In many species and grades, the K value exceeds 20, and formula (36) may be used for intermediate columns coming under the Forest Products Laboratory fourth-power parabolic formula⁷ instead of the Euler formula. Where this is the case, the value of c in equation (36) is determined from the parabolic instead of the Euler formula, but use of equation (36) is otherwise the same.

For columns with slenderness $(\frac{L}{d})$ ratios between 11 and 20, the safe loadings under side load and eccentricity may be determined by straight-line interpolation between equation (9) for $\frac{L}{d} = 11$ and equation (36) for $\frac{L}{d} = 20$. Loadings thus determined may be somewhat in error on the conservative side.

Examples

Assume a species and grade of wood having the following properties:

E = 1,600,000 pounds per square inch
f = 1,600 pounds per square inch
c = 1,200 pounds per square inch (for short column)

Values of c for intermediate or long columns are obtained with the Wood Handbook formulas.²

Assume a 6- by 8-inch (nominal dimensions) column in various lengths and under various conditions of loading. The side loads and eccentricity are assumed as acting to bend the column in the 8-inch direction. The actual cross-sectional dimensions are 5-1/2 by 7-1/2 inches.

The design of the column in these examples provides only for flexure in the 8-inch direction. The column is assumed to be stable or supported so that there is no deflection in the 6-inch direction; if simultaneous deflection in both directions is possible, the analysis becomes more complex. For the purpose of determining limits of application of the formulas in the examples, the slenderness ratio is based on 7-1/2 rather than 5-1/2 inches.

Long Column with Side Load and Concentric Axial Load

Let the column be 20 feet long, so that $\frac{L}{d} = 32$. Assume a side load of 75 pounds per foot of length, uniformly distributed. Determine the safe axial load centrally applied.

Equation (36) is utilized by dropping the terms representing eccentricity and side load proportional to end load, so that it becomes

$$\frac{M/S}{f - P/A} + \frac{P/A}{c} = 1$$

Since M/S is known, the equation is solved for P/A giving

$$P/A = \left(\frac{f + c}{2}\right) - \sqrt{\left(\frac{f + c}{2}\right)^2 - c(f - M/S)} \quad (37)$$

$$M/S = \frac{75 \times 20 \times 20 \times 12 \times 6 \times 4 \times 2}{8 \times 11 \times 15 \times 15} = 873 \text{ p.s.i.}$$

² Forest Products Laboratory Wood Handbook. U. S. Dept. of Agr., revised 1955) p. 216.

$$c = \frac{0.274 \times 1,600,000}{32 \times 32} = 428 \text{ p.s.i.}$$

$$\frac{f + c}{2} = \frac{1,600 + 428}{2} = 1,014$$

$$P/A = 1,014 - \sqrt{(1,014)^2 - 428(1,600 - 873)} = 167 \text{ p.s.i.}$$

The safe load is $5.5 \times 7.5 \times 167 = 6,900$ pounds.

Eccentric Axial Load on Long Column

Let the column be 20 feet long, and the eccentricity be 2-1/2 inches, with no side load.

Equation (36) takes the form

$$\frac{P/A \left(\frac{15e}{2d} \right)}{f - P/A} + \frac{P/A}{c} = 1$$

which, solved for P/A gives

$$P/A = \frac{f + c \left(1 + \frac{15e}{2d} \right)}{2} - \sqrt{\left[\frac{f + c \left(1 + \frac{15e}{2d} \right)}{2} \right]^2 - cf} \quad (38)$$

$f = 1,600$ p.s.i. and $c = 428$ p.s.i. as before

$$\left(1 + \frac{15e}{2d} \right) = 1 + \frac{15 \times 5 \times 2}{2 \times 15 \times 2} = 3.5$$

$$\frac{f + 3.5c}{2} = 1,549$$

$$P/A = 1,549 - \sqrt{(1,549)^2 - (428)(1,600)} = 240 \text{ p.s.i.}$$

The safe load is $5.5 \times 7.5 \times 240 = 9,900$ pounds.

If the length of the column is 12-1/2 feet, so that $\frac{L}{d} = 20$, the solution is the same except for the value of c . Since $K = 23.4$, which is greater than 20, the fourth-power parabolic formula is used, and $c = 986$ pounds per square inch.

Substitution of this value in equation (38) gives $\frac{P}{A} = 335$ pounds per square inch. The safe load is $5.5 \times 7.5 \times 335 = 13,800$ pounds.

Eccentric Axial Load on Short Column

Let the column be 7 feet long, so that $\frac{L}{d} = 11$. The eccentricity is 2-1/2 inches and there is no side load. Determine the safe load.

Equation (8) is
$$\frac{P/A \left(\frac{6e}{d}\right)}{f} + \frac{P/A}{c} = 1.$$

When solved for P/A , this becomes

$$P/A = \frac{cf}{f + c \left(\frac{6e}{d}\right)} \quad (39)$$

$$c = 1,200 \text{ p.s.i.}, f = 1,600 \text{ p.s.i.}, \frac{e}{d} = \frac{5 \times 2}{15 \times 2} = \frac{1}{3}$$

$$P/A = \frac{1,200 \times 1,600}{1,600 + 2,400} = 480 \text{ pounds per square inch}$$

The safe load is $5.5 \times 7.5 \times 480 = 19,800$ pounds.

Eccentric Axial Load on Intermediate Column

If the column is 10 feet long, so that $\frac{L}{d} = 16$, the safe load is obtained by straight-line interpolation between the values for $\frac{L}{d} = 11$ and $\frac{L}{d} = 20$.

$$19,800 - 5/9 (19,800 - 13,800), = 16,500 \text{ pounds.}$$

Axial Load Known

Where $\frac{P}{A}$ has a known value and it is desired to determine the permissible side load, equations (9) or (36) or any modifications thereof can be solved for $\frac{M}{S}$. Since $\frac{M}{S}$ appears only in the first power in either equation, solution for it offers no special problems. Having determined $\frac{M}{S}$, the permissible side load for the assumed conditions can be found.

Eccentricity and Two Side Loads

A 4- by 16-inch (nominal) member is used with width vertical in the upper chord of a roof truss. It supports a number of uniformly distributed purlin loads, with purlin assumed to give full lateral support so that the member qualifies as a short column. The roof system is designed so that the flexural stress in the member from the roof loads through the purlins is one-half the axial stress in the member from the truss reactions. There is also a concentrated load suspended near the center of length of the member, causing a flexural stress of 200 pounds per square inch. The design at the panel points of the truss is such that the axial load is centered at a point 1.55 inches above the center of width of member. Physical properties of the material are the same as in preceding examples. Determine the safe axial load, assuming actual dimensions of 3-5/8 by 15-1/2 inches.

In this case, equation (9) in its complete form is used. When solved for P/A , this becomes

$$P/A = \frac{c (f - M/S)}{f + c \left(\frac{6e}{d} + z \right)} \quad (40)$$

$$\frac{e}{d} = \frac{1.55}{15.5} = \frac{1}{10} \quad z = \frac{1}{2}$$

$$P/A = \frac{1,200 (1,600 - 200)}{1,600 + 1,200 \left(\frac{6}{10} + \frac{1}{2} \right)} = \frac{1,200 \times 1,400}{1,600 + 1,320} = 525 \text{ p.s.i.}$$

$$z P/A = \frac{525}{2} = 262 \text{ pounds per square inch flexural stress from purlin loads.}$$

Safe axial load is $525 \times 3\text{-}5/8 \times 15\text{-}1/2 = 29,500$ pounds.

Column Formula Summarized

The general formulas developed by Newlin are

$$\frac{P/A \left(\frac{6e}{d} \right) + M/S + z P/A}{f} + \frac{P/A}{c} = 1 \quad (9)$$

for columns with slenderness ratios of 11 or less, and

$$\frac{P/A \left(\frac{15e}{2d} \right) + M/S + z P/A}{f - P/A} + \frac{P/A}{c} = 1 \quad (36)$$

for columns with slenderness ratios of 20 or more, where

P/A = direct compressive stress induced by axial load.

M/S = flexural stress induced by side loads that are independent of end load.

c = allowable unit working stress in compression parallel to grain for a column of the slenderness ratio under consideration with centrally applied axial load and no side load.

f = allowable unit working stress in flexure that is permitted where flexural stress only exists.

e = eccentricity.

d = depth of column, measured in the direction of side loads or eccentricity.

z = ratio of flexural to direct compressive stress when both result from the same loading, so that the ratio remains constant while the load varies.

Stresses for columns with slenderness ratios between 11 and 20 are determined by straight-line interpolation between formula (9) for a slenderness ratio of 11 and formula (36) for a slenderness ratio of 20.

These formulas may be simplified for some conditions of loading by dropping out certain terms; for example, if there are side loads and a concentrically applied end load, e becomes zero, and the first term in the numerator of equations (9) and (36) disappears. The formulas can be solved for P/A or for M/S where this will facilitate their use.

These formulae are applicable only to columns of square or rectangular cross section.

Where side loads are such that maximum deflection axial maximum flexural stress do not occur at midlength of the column, it is generally satisfactory to consider M/S as the maximum flexural stress due to the load or loads, regardless of its position in the length of the column. When the point of this maximum stress is near the end of the column, a slight error on the side of overload will occur.

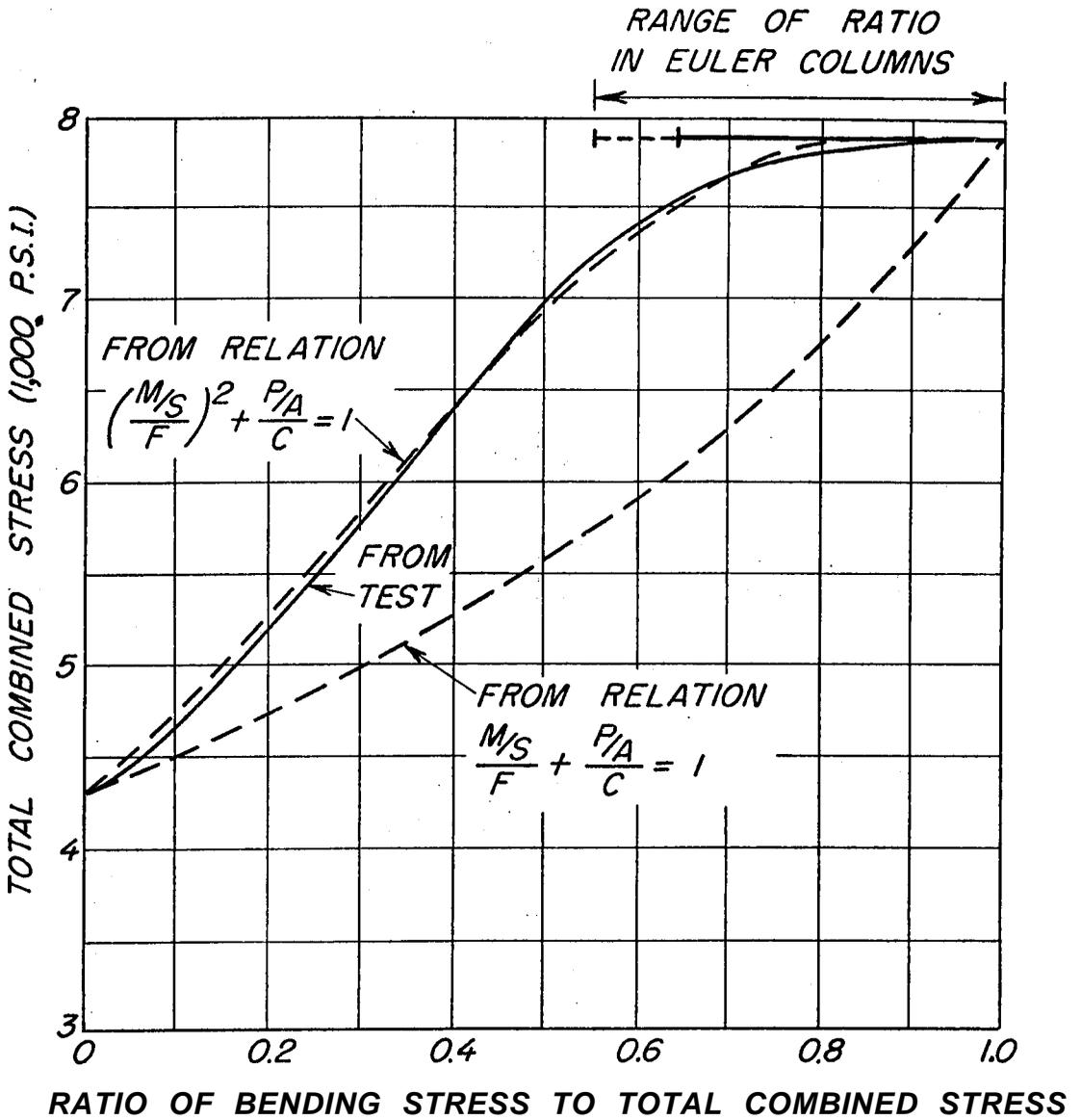


Figure 1.--Strength values of clear Sitka spruce under combined stress, as determined by test and calculated by two methods.

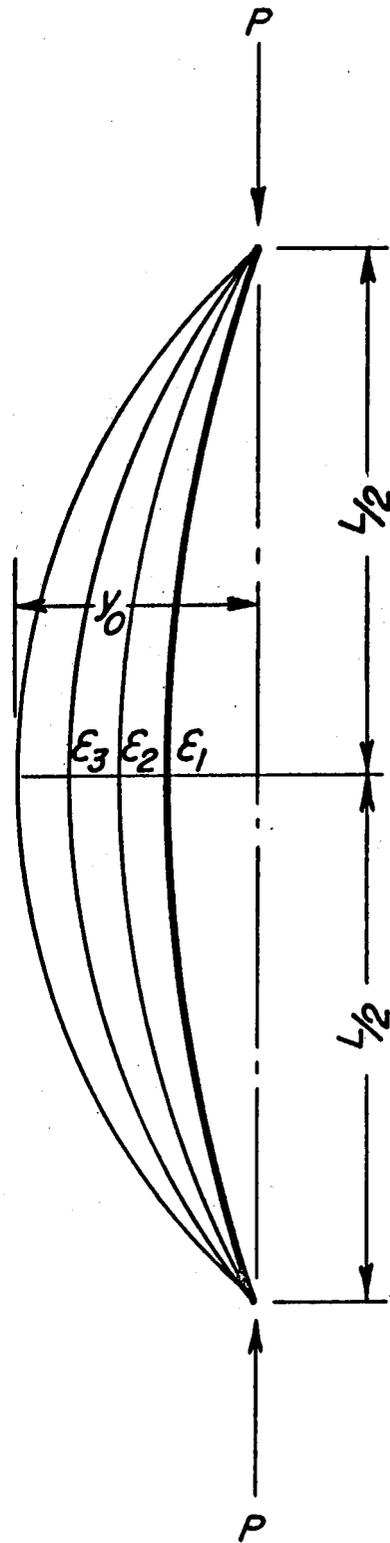


Figure 2.--Load on column with sinusoidal curvature.

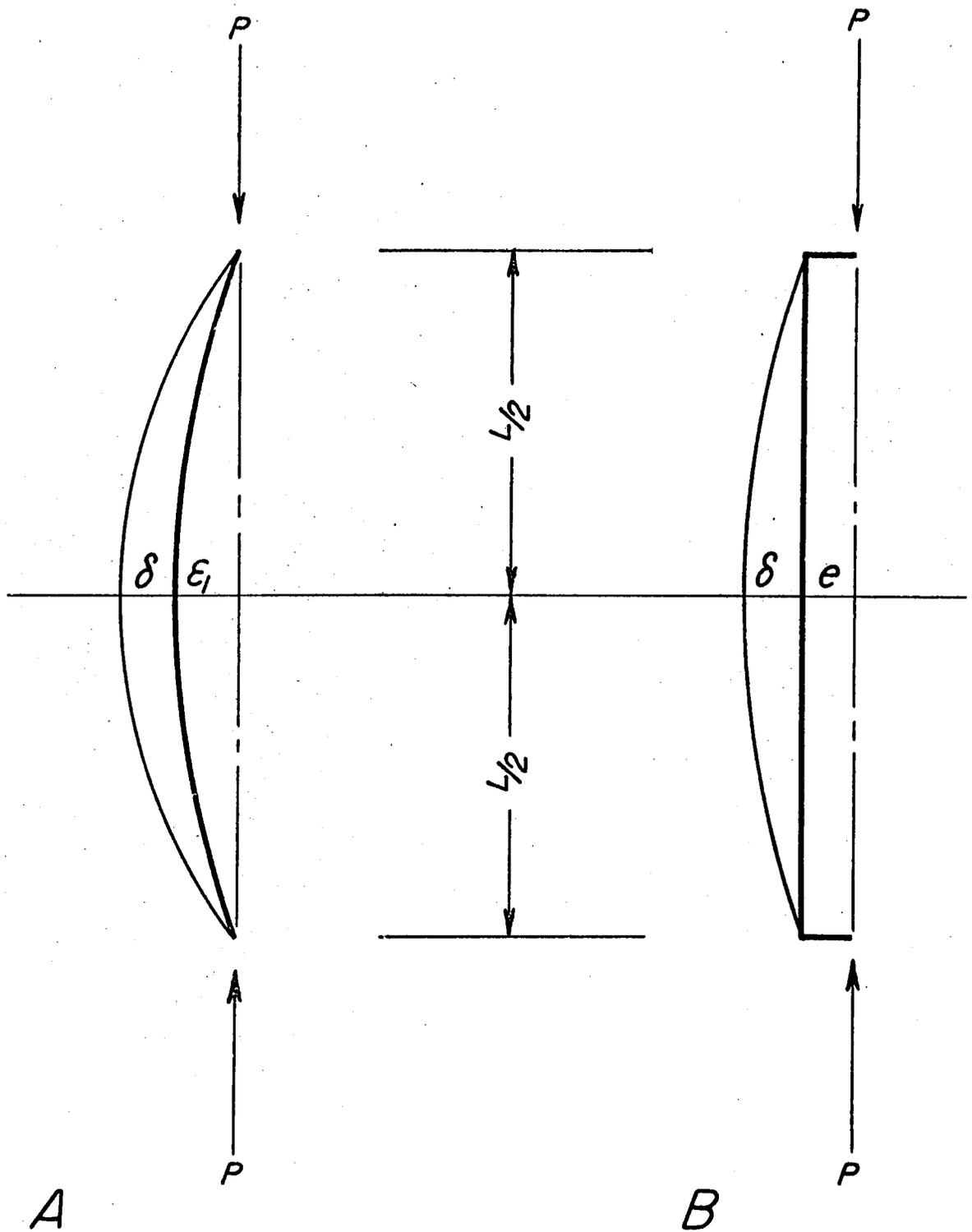


Figure 3.--Straight column with eccentricity equivalent to Salmon's originally bent column, A, Salmon's originally bent column; B, straight column with eccentric loading.

SIDE LOADS AND ECCENTRICITIES ACTING IN TWO DIRECTIONS ON COLUMNS

Supplement to

FORMULAS FOR COLUMNS WITH SIDE LOADS AND ECCENTRICITY

By

C. B. NORRIS, Engineer
and
LYMAN W. WOOD, Engineer

October 1951

This supplement presents formulas for columns with stresses and deflections resulting from side loads or eccentricities in more than one direction. The concept is thus broadened from that of the main body of this report, which deals with side loads and eccentricities in the same plane and assumes that deflection of the column takes place only in that plane.

A solution of this problem for short columns is not difficult, since it is necessary only to sum up the stresses without taking into account the deflections. Consider equation (9), using the same notation for stresses and eccentricities in one direction and an equivalent notation with prime superscripts for the counterparts in a direction at right angles. Equation (9) then becomes

$$\frac{\frac{P}{A} \left(\frac{6e}{d} + \frac{6e'}{d'} \right) + \frac{M}{S} + \frac{M'}{S'} + \frac{P}{A} (z + z')}{f} + \frac{P}{c} = 1 \quad (9')$$

Note in the above that P is the full compressive load, causing bending stresses from eccentricity in both directions in addition to the direct compression. Equation (9') defines the condition for maximum allowable stress in the extreme fiber, which in this case is on a corner rather than along one side of the column.

The problem in long columns is somewhat more complex, since the deflections must be considered, but an analysis along lines similar to that for short columns seems possible.

Equation (17) has for its counterpart

$$\frac{\epsilon_1' + \epsilon_2' + \epsilon_3'}{y_0'} + \frac{P}{W'} = 1 \quad (17')$$

representing deflections at right angles to those of equation (17). Note here that W' the critical Euler load, differs from W except for columns of square cross section, where the two are the same.

Substitution of stresses for deflections gives a counterpart to equation (18)

$$\frac{f_1' + f_2' + f_3'}{f_0'} + \frac{P}{Aw'} = 1 \quad (18')$$

Substitution of equivalent stresses gives a counterpart to equation (33)

$$\frac{w' \left(\frac{15e'}{2d'} \right) + \frac{M'}{S'} + z' \frac{P}{A}}{f_0'} + \frac{P}{Aw'} = 1 \quad (33')$$

Equations (33) and (33') are solved for f_0 and f_0' , respectively, the stresses corresponding to the total deflections when all eccentric and side loads are on. This gives

$$f_0 = \frac{w \frac{15e}{2d} + \frac{M}{S} + z \frac{P}{A}}{1 - \frac{P}{Aw}} \quad (33)$$

and

$$f_0' = \frac{w' \frac{15e'}{2d'} + \frac{M'}{S'} + z' \frac{P}{A}}{1 - \frac{P}{Aw'}} \quad (33')$$

Let f_t be a total stress corresponding to the total of all deflections in both directions. For safe loadings, use \underline{c} and \underline{c}^1 instead of w and w^1 (in square columns, $\underline{c} = \underline{c}^1$). Then

$$f_t = f_o + f_o' = \frac{c \frac{15e}{2d} + \frac{M}{S} + z \frac{P}{A}}{1 - \frac{P}{Ac}} + \frac{c' \frac{15e'}{2d'} + \frac{M'}{S'} + z' \frac{P}{A}}{1 - \frac{P}{Ac'}} \quad (37)$$

By a reasoning analogous to that leading to equation (35), the safe stress \underline{f} must be equal to

$$f_t + \frac{P}{A} - c \frac{15e}{2d} - c' \frac{15e'}{2d'}$$

or by transposition

$$f_t = f - \frac{P}{A} + c \frac{15e}{2d} + c' \frac{15e'}{2d'} \quad (38)$$

Elimination of \underline{f}_t between equations (37) and (38) gives

$$\frac{c \frac{15e}{2d} + \frac{M}{S} + z \frac{P}{A}}{1 - \frac{P}{Ac}} + \frac{c' \frac{15e'}{2d'} + \frac{M'}{S'} + z' \frac{P}{A}}{1 - \frac{P}{Ac'}} = f - \frac{P}{A} + c \frac{15e}{2d} + c' \frac{15e'}{2d'} \quad (39)$$

a counterpart of equation (35).

Algebraic transformation of equation (39) by clearing fractions and collecting terms gives the form

$$\begin{aligned} \frac{M}{S} + \frac{M'}{S'} + \frac{P}{A} (z + z' - \frac{M}{Sc'} - \frac{M'}{S'c} + \frac{15e}{2d} + \frac{15e'}{2d'}) - \left(\frac{P}{A}\right)^2 \left(\frac{z}{c'} + \frac{z'}{c} + \frac{15e}{2dc'} + \frac{15e'}{2d'c}\right) \\ = \left(f - \frac{P}{A}\right) \left(1 - \frac{P}{Ac}\right) \left(1 - \frac{P}{Ac'}\right) \end{aligned} \quad (40)$$

Equation (40) defines the condition for safe loading. It is somewhat complex, but in its general form, any combination of side loads and eccentricities can be represented by it. It may be simplified for some conditions; for example, \underline{z} and \underline{z}' may be equal or one or both may be zero; or in square column, $\underline{c} = \underline{c}'$ and $\underline{d} = \underline{d}'$.

Equation (40) is a cubic equation in $\underline{P/A}$ so that, with other quantities known, three possible values for $\underline{P/A}$ are indicated. Of the three, the least value that that is not imaginary should be used in design. Solution for

$\frac{M}{S}$ or $\frac{M'}{S'}$ is more simple and yields only one value.