STRENGTH CRITERIA FOR ORTHOTROPIC MATERIALS

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This paper discusses the three strength criteria for orthotropic materials: the Hankinson formula, the Norris theory, and the Tsai–Hill theory.

Hankinson Formula

The most common orthotropic material is wood. In 1921, Hankinson [1] proposed the first formula for the compressive strength of wood in a principal material plane:

$$F_x = \frac{F_1 F_2}{F_1 \sin^n \theta + F_2 \cos^n \theta} \tag{1}$$

where F_1 and F_2 are the compressive strengths in axis 1 (the grain direction) and axis 2 (a direction perpendicular to the grain), F_x is the compressive strength in the x direction at an angle q from the 1 axis in the 1–2 plane, and n = 2. Later, other researchers found that n may vary between 2 and 2.5. Eq. (1), being strictly empirical, is also suitable for computing the tensile strength of wood (Fig. 1) when F_1 and F_2 are tensile and n varies between 1.5 and 2 [2].

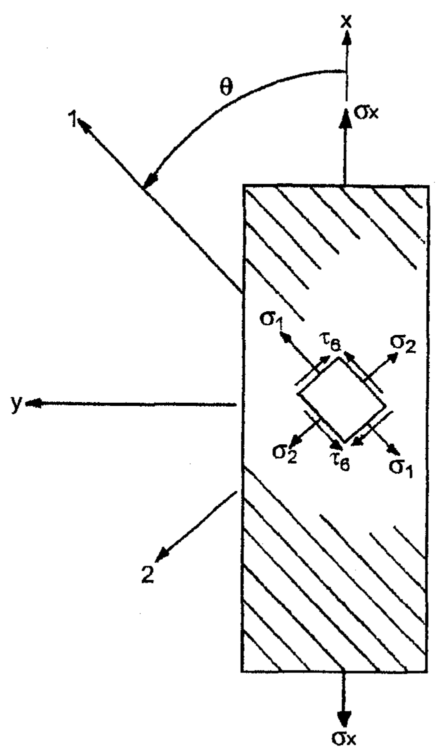

Figure 1. Schematic of off-axis tension specimen (x and y are geometrical axes, 1 and 2 are material axes),

Norris Theory

In 1955, Norris [4] developed a theory for the strength of orthotropic materials based on the Henky–von Mises theory for isotropic materials. He considered an orthotropic material to be made up of an isotropic material by introducing voids in the shape of equal rectangular prisms. The walls of isotropic material between these voids form the three principal planes of the orthotropic material. Using the energy of distortion expression, he obtained a formula for each of these planes. For the 1–2 plane,

$$\frac{\sigma_1^2}{F_1^2} + \frac{\sigma_2^2}{F_2^2} + \frac{\tau_6^2}{F_6^2} - \frac{\sigma_1 \sigma_2}{F_1 F_2} = 1 \tag{2}$$

Consider the case of uniaxial off-axis loading shown in Fig. 1. By transforming the applied stress s$_x$ along the principal material axes,

$$\sigma_1 = \sigma_x \cos^2 \theta, \qquad \sigma_2 = \sigma_x \sin^2 \theta, \\ \tau_6 = -\sigma_x \sin\theta \cos\theta \tag{3}$$

and substituting into Eq. (2), we obtain the equation for axial strength F_x (max s$_x$ = F_x):

$$\frac{1}{F^2{}_x} = \frac{\cos^4 \theta}{F_1^2} + \frac{\sin^4 \theta}{F_2{}^2} + \left(\frac{1}{F_6^2} - \frac{1}{F_1 F_2}\right) \sin^2 \theta \cos^2 \theta \tag{4}$$

If for a given value of F_6 in Eq. (4) there exists a fixed value of n in Eq. (1) (other parameters being the same), we conclude that n provides the information about F_6.

Tsai–Hill Theory

Of all the macromechanical failure theories for anisotropic materials, the Tsai–Hill theory and the Tsai–Wu theory are the most widely used [6]. The Tsai–Wu theory requires both tensile and compressive strength data as input. When only one is available, the Tsai–Hill theory is used. The Tsai–Hill theory is identical to Eq. (2), except where F_1F_2 appears, it is replaced by F_1^2. The theory was derived following the undetermined coefficients method. The F_1^2 was obtained using an imaginary biaxial test without considering the interaction of the applied loading [6]. For the same case in Fig. 1, the Tsai–Hill theory becomes

$$\frac{1}{F_x^2} = \frac{\cos^4\theta}{F_1^2} + \frac{\sin^4\theta}{F_2^2} + \left(\frac{1}{F_6^2} - \frac{1}{F_1^2}\right)\sin^2\theta\cos^2\theta \qquad (5)$$

Numerical Results

Fig. 2 presents the tensile strength results of Eq. (4) with F_1 = 78.3 MPa, F_2 = 2.55 MPa, and F_6 = 7.93 MPa for Sitka spruce [3]. If we assume n = 1.97 while F_1 and F_2 remain the same in Eq. (1), we obtain very similar results. Taking F_6 = 6.25 MPa [5] with the same F_1 and F_2, the results from Eq. (4) are also plotted in Fig. 2. For this case, with n = 1.78, q. (1) can produce a close match. Results in Fig. 2 indicate that the application of elasticity theory in wood at failure is *approximately* correct.

Fig. 3 compares the results of the Norris theory with the Tsai–Hill theory for the same data. Referring to the numerical comparison between the Tsai–Wu and Tsai–Hill theories on uniaxial strength of off-axis E-glass/epoxy lamina versus fiber orientation given by Daniel and Ishai [6], we see that the Tsai–Wu solution is above the Tsai–Hill solution just as the Norris solution is above the Tsai–Hill solution in Fig. 3.

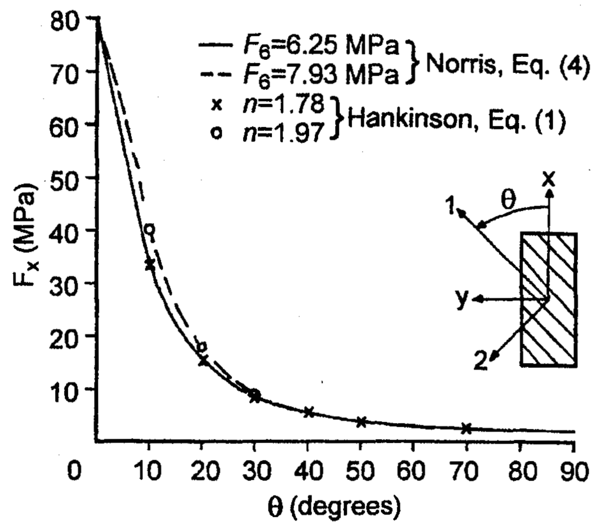

Figure 2. Failure curves based on Norris theory and Hankinson formula (F_1= 78.3 MPa, F_2 = 2.55 MPa).

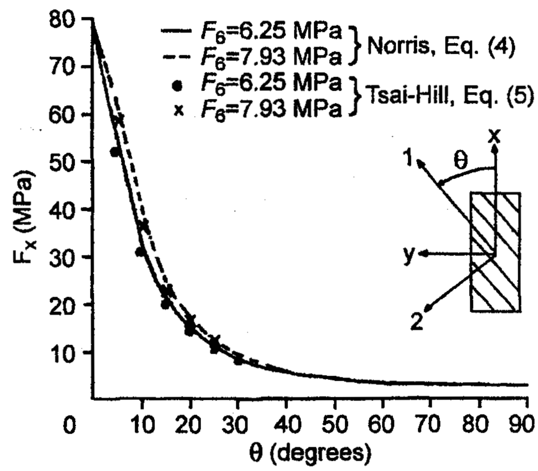

Figure 3. Failure curves based on Norris theory and Tsai–Hill theory (F_1 = 78.3 MPa, F_2 = 2.55 MPa).

References

1. Hankinson, R.L. 1921. Investigation of Crushing Strength of Spruce at Varying Angles of Grain. Air Service Information Circular No.259, U.S. Air Service.

2. FPL. 1999. *Wood Handbook*. Gen Tech Rep.FPL–GRT–113, p. 4-29, Madison, WI: U.S Department of Agriculture, Forest Products Laboratory, 463 p.

3. Green, D. W. 2000. Private Communication.

4. Norris, C.B. 1955. Strength of Orthotropic Materials Subjected to Combined Stresses. U.S. Forest Products Lab. Rep. 1816, FPL, Madison, WI.

5. Liu, J.Y. and Floeter, L.H. 1984. Shear Strength in Principal Plane of Wood. Journal of Engineering Mech. 110(6):930-936.

6. Daniel, I.M., and Ishai, O. 1994. *Engineering Mechanics of Composite Materials*. Oxford, 395 p.

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