EFFECT OF IN-PLANE SHEAR MODULUS OF ELASTICITY ON BUCKLING STRENGTH OF PAPERBOARD PLATES[1]

Thomas J. Urbanik

Research General Engineer
USDA Forest Service
Forest Products Laboratory
One Gifford Pinchot Drive
Madison, WI 53705-2398

(Received August 1991)

ABSTRACT

In previous research, a thin-plate theory was derived for analyzing corrugated fiberboard under edgewise compression and subjected to localized buckling. In this note, buckling formulas for thin paperboard plates characterized by an approximate in-plane shear modulus of elasticity are further generalized to allow for arbitrary levels of shear modulus. The results have applications in the design of paper products made of composite plates.

Keywords: Plates, elastic stability, buckling, shear modulus, paper.

INTRODUCTION

Previous research derived the theory appropriate to analyzing corrugated fiberboard under edgewise compression and subjected to localized buckling. The thin-plate theory of Johnson and Urbanik (1987) considered the elastic buckling of long rectangular plate elements of paper under longitudinal compressive loading. The thin plate theory was later incorporated in a theory of plate structures applicable to modeling corrugated fiberboard short-column structures (Johnson and Urbanik 1989). Basic stress-strain inputs to the analysis are the combination of properties c_1, $c_2 = E_2$, ν_1/ν_2, and $\nu = \sqrt{\nu_1\nu_2}$ for each plate element.

Johnson and Urbanik (1987) assumed that the plate material is homogeneous and that the initial in-plane shear modulus of elasticity G could be approximated by a relationship due to St. Venant. As a result, instead of G being an independent input to the structure buckling analysis of Johnson and Urbanik (1989), G is made a function of other stress-strain constants according to

$$G = \frac{E_2\sqrt{\nu_1/\nu_2}}{2(1 + \sqrt{\nu_1\nu_2})} \qquad (1)$$

The approximation made sense in that paper shear modulus is difficult to determine.

In analyzing steel structures, the procedure for conceptually defining the elastic rigidities of a plate-stiffener combination with those of an equivalent constant thickness plate—successfully applied by Huffington (1956)—reduces to determining four independent rigidities of the equivalent plate. Likewise, the theory can have further applications to corrugated containers, spaceboard (Setterholm 1985), and other nonhomogeneous fiber products if effective stress-strain properties characterizing the elastic constants of the panel material can be determined.

The objective of this note is to extend the results of Johnson and Urbanik (1987) and set forth the equations so that in applying the theory to other than thin paper materials the effect of plate shear modulus on structure buckling strength can be considered. Because this note supplements earlier work, the complete deri-

[1] This article was written and prepared by U.S. Government employees on official time, and it is therefore in the public domain and not subject to copyright. The Forest Products Laboratory is maintained in cooperation with the University of Wisconsin.

TABLE 1. *Forms of equations expanded to include in-plane shear.*

Equation number[a]	Generalized equation form	New equation number
—	$\hat{c} = \nu + 2(1 - \nu^2)\frac{G}{c_2}\sqrt{\frac{\nu_2}{\nu_1}}$	2.10′
2.11	$\hat{\beta} = \chi\left\{-\hat{c} + \sqrt{1-\nu^2}\left[\frac{\hat{c}^2 - 1}{1-\nu^2} + f(\hat{\epsilon}) + \frac{3\hat{\epsilon}}{\chi^2 S}\right]^{1/2}\right\}^{1/2}$	2.11′
2.12	$\hat{\alpha} = (\hat{\beta}^2 + 2\chi^2\hat{c})^{1/2}$	2.12′
2.13	$\hat{\epsilon} = \frac{\chi^2 S}{3}\left[\frac{1}{1-\nu^2}\left(\hat{c} + \frac{\hat{\beta}^2}{\chi^2}\right)^2 - \frac{\hat{c}^2 - 1}{1-\nu^2} - f(\hat{\epsilon})\right]$	2.13′
3.2	$\hat{\epsilon} = \frac{\chi^2 S}{3}\left[\frac{1}{1-\nu^2}\left(\hat{c} + \frac{\pi^2}{4\chi^2}\right)^2 - \frac{\hat{c}^2 - 1}{1-\nu^2} - f(\hat{\epsilon})\right]$	3.2′
3.4	$\hat{\epsilon} = \frac{\pi^2 S}{6(1-\nu^2)}\{\hat{c} + [1 - (1-\nu^2)f(\hat{\epsilon})]^{1/2}\}$	3.4′
3.5	Initial $\hat{\epsilon} = \frac{\pi^2 S(\hat{c}^2 + 1)^2}{12(1-\nu^2)}$	3.5′
4.4	$\hat{\beta} = \tan^{-1}[-(\hat{\beta}^2 + 2\chi^2\hat{c})^{1/2}\tanh(\hat{\beta}^2 + 2\chi^2\hat{c})^{1/2}/\hat{\beta}] + \pi$	4.4′
4.5	$\chi = \left[\frac{\hat{\beta}^4 - 2\hat{c}\hat{\beta}\hat{\beta}'\chi^4}{2\hat{\beta}^3\hat{\beta}' - \hat{c}\hat{\beta}^2 + 1.5(1-\nu^2)\hat{\epsilon}/S}\right]^{1/2}$	4.5′
5.3	$C = \frac{\chi^2}{3}\left[\frac{1}{1-\nu^2}\left(\hat{c} + \frac{\hat{\beta}^2}{\chi^2}\right)^2 - \frac{\hat{c}^2 - 1}{1-\nu^2}\right]$	5.3′
—	$C = \frac{\pi^2(\hat{c} + 1)^2}{12(1-\nu^2)}$	5.3.1′

[a] From Johnson and Urbanik (1987).

vations and definitions of terms leading to the results are not repeated. Unless otherwise indicated, equation numbers used within the text of this note refer to the equations of Johnson and Urbanik (1987); equation numbers designated with a prime (′) refer to their more generalized form.

THEORY

The first equation from the set of Eq. (2.8) (Johnson and Urbanik 1987) is retained in the form

$$\frac{2H_{12} + H_{33}}{H_{11}} = \nu_2(2 + c) \tag{2}$$

Equation (2.11) is then derived as follows: The more generalized form for $\hat{\beta}$ becomes

$$\hat{\beta} = \chi\left\{\frac{-(2+c)\nu}{2} + \sqrt{1-\nu^2}\left[\frac{[(2+c)\nu/2]^2 - 1}{1-\nu^2} + f(\hat{\epsilon}) + \frac{3\hat{\epsilon}}{\chi^2 S}\right]^{1/2}\right\}^{1/2} \tag{3}$$

The new dimensionless shear constant is then introduced as

$$\hat{c} = \frac{(2+c)\nu}{2} \tag{4}$$

and instead of using Eq. (1.4), a more general expression equal to Eq. (3.3) of Johnson and Urbanik (1984) is substituted for c:

TABLE 2. *Stress-strain properties for four materials and dimensionless buckling stress when $S = 0.3$.*

Material	E_1 (GPa)	E_2 (GPa)	ν_1	ν_2	G (GPa)	$\hat{c}$	Simple $\hat{\sigma}$	Difference (%)
Steel	207	207	0.303	0.303	78.7	0.99	0.75	0
						1	0.75	
Paper	6.87	3.20	0.403	0.135	1.69	0.81	0.69	5.6
						1	0.73	
Plate 1	131	165	0.280	0.221	47.2	0.85	0.70	4.3
						1	0.73	
Plate 2	55.8	111	0.437	0.219	34.3	1.1	0.77	−2.5
						1	0.75	

$$c = \frac{4(1 - \nu_1\nu_2)}{\nu_1 E_2} G \qquad (5)$$

The result is the expression given by Eq. (2.10′) in Table 1. Other equations requiring further generalizations are also listed there. Letting $\hat{c} = 1$ is equivalent to assuming the St. Venant approximation, and doing so reduces the generalized equations to their forms given by Johnson and Urbanik (1987).

FIXED-PLATE ALGORITHM

In determining the buckling strain of a fixed-edge plate, the algorithm given by Johnson and Urbanik (1987) sometimes fails to converge when $\hat{c} \neq 1$. A new algorithm, motivated by Fig. 5 of Johnson and Urbanik (1987), makes use of the fact that in the vicinity of point P_2, which defines the minimum $\hat{\epsilon}$ at the optimum χ, the form of the curve Y_1Y_2 can be determined from the interpolating polynomial approximation to Eq. (2.13′) given by

$$\hat{\epsilon}(\chi) = b_1\chi^2 + \frac{b_2}{\chi^2} + b_3 \qquad (6)$$

A relative minimum occurs when $d\hat{\epsilon}/d\chi = 0$ at

$$\chi = \left(\frac{b_2}{b_1}\right)^{1/4} \qquad (7)$$

Therefore, the estimate of an optimum χ is updated from the values of b_1 and b_2 determined by fitting $\hat{\epsilon}(\chi)$ to each three most recent pairs of $\hat{\epsilon}$-χ estimates.

Note from Eq. (2.11′) (Table 1) that low values of G input to Eq. (2.10′) may cause the expression within brackets to become negative. This complicates the plate structure buckling analysis of Johnson and Urbanik (1989). If the lowest singularity among the plates is used to upwardly bound the search for a root at a fixed λ, the search may seek out complex roots. The form of the complex region appears in Fig. 5 of Johnson and Urbanik (1987) as the region below and to the right of line R_1-R_2. The program for analyzing a plate system was therefore augmented by an additional subprogram that constrains the search for an optimum λ [Eq. (4.3)] of Johnson and Urbanik (1989) to the real region.

RESULTS AND CONCLUSIONS

Experimental data on steel from Huffington (1956) and on paper from Suhling et al. (1989) (summarized in Table 2) can be used to test the severity of the St. Venant approximation. The paper data are for a commercial 205 g/m^2 linerboard. The plate data are effective stress-strain properties determined by assuming that the effective plate thickness equals the original thickness and equating experimental rigidities to the values predicted by bending moduli H_{11}, H_{22}, H_{12}, and H_{33}. Plate 1 is a steel plate with parallel ribs machined on one face by removing 16.2% of the material to a depth of 25% of the plate thickness. Plate 2 has its second face equally machined. For comparison, data on mild steel are also given.

Figure 1 shows how plate buckling stress would vary with plate stiffness according to the generalized theory of Johnson and Urbanik

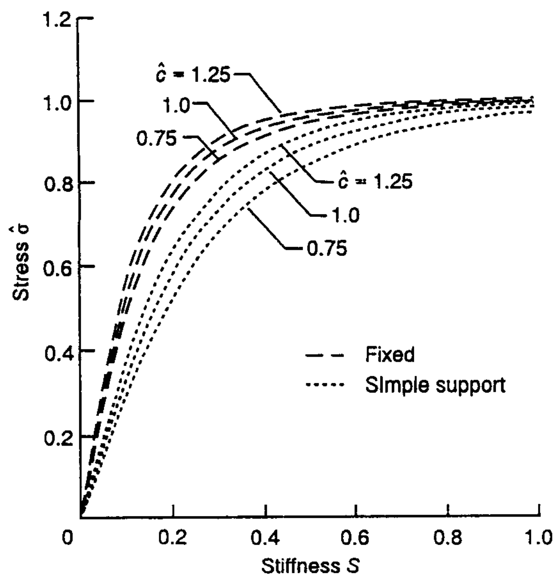

FIG. 1. Plots of $\hat{\sigma}$ as a function of S for three levels of $\hat{c}$, two support conditions, and $\nu = 0.25$.

(1987). More specific comparisons are made in Table 2. Dimensionless buckling stress of a simple support plate is predicted for the case $S = 0.3$ using the $\hat{c}$ value predicted from data for comparison with $\hat{c} = 1$. For the limited data, using an approximate shear modulus seems to be sufficient. When the shear modulus is known, the generalized equations from Table 1 can be used to improve the buckling strength predictions.

NOMENCLATURE

b_1, b_2, b_3	Regression constants
C	Linear buckling constant
c	Shear constant
c_1, c_2	Constants in uniaxial stress-strain curve $\sigma = c_1 \tanh(c_2\epsilon/c_1)$
$\hat{c}$	Dimensionless shear constant
E_1, E_2	Young's modulus in 1-direction and loaded 2-direction
f	Function of $\hat{\epsilon}$
G	Shear modulus of elasticity
$H_{11}, H_{22}, H_{12}, H_{33}$	Plate bending stiffness moduli
P_2, R_1, R_2, Y_1, Y_2	Points on graph
S	Dimensionless plate stiffness
$\hat{\alpha}$	Dimensionless root of buckling equation
$\hat{\beta}$	Dimensionless root of buckling equation
$\hat{\beta}'$	Partial derivative of $\hat{\beta}$
$\hat{\epsilon}$	Normalized buckling strain
λ	Buckling wave periodicity
$\hat{\sigma}$	Dimensionless buckling stress
χ	Dimensionless periodicity of buckling wave
ν	Geometric mean of Poisson's ratio
ν_1, ν_2	Poisson's ratios in 1-direction and loaded 2-direction

REFERENCES

HUFFINGTON, N. J., JR. 1956. Theoretical determination of rigidity properties of orthogonally stiffened plates. J. Appl. Mech. March:15–20.

JOHNSON, M. W., JR., AND T. J. URBANIK. 1984. A nonlinear theory for elastic plates with applications to characterizing paper properties. J. Appl. Mech. 51(3):145–152.

———, AND ———. 1987. Buckling of axially loaded, long rectangular paperboard plates. Wood Fiber Sci. 19(2):135–146.

———, AND ———. 1989. Analysis of the localized buckling in composite plate structures with application to determining the strength of corrugated fiberboard. ASTM J. Composites Technol. & Res. 11(4):121–127.

SETTERHOLM, V. C. 1985. FPL spaceboard—A new structural sandwich concept. Tappi J. 66(6):40–42.

SUHLING, J. C., D. E. GUNDERSON, M. W. JOHNSON, AND R. E. ROWLANDS. 1989. Nonlinear elastic constitutive relations for cellulosic materials. *In* Proceedings, 3rd joint ASCE/ASME Mechanics Conf., July 1–15 1989.