MULTIVARIATE STOCHASTIC SIMULATION

WITH SUBJECTIVE MULTIVARIATE NORMAL DISTRIBUTIONS[1]

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Abstract.-In many applications of Monte Carlo simulation in forestry or forest products, it may be known that some variables are correlated. However, for simplicity, in most simulations it has been assumed that random variables are independently distributed. This report describes an alternative Monte Carlo simulation technique for subjectively assessed multivariate normal distributions. The method requires subjective estimates of the 99-percent confidence interval for the expected value of each random variable and of the partial correlations among the variables. The technique can be used to generate pseudorandom data corresponding to the specified distribution. If the subjective parameters do not yield a positive definite covariance matrix, the technique determines minimal adjustments in variance assumptions needed to restore positive definiteness. The method is validated and then applied to a capital investment simulation for a new papermaking technology. In that example, with ten correlated random variables, no significant difference was detected between multivariate stochastic simulation results and results that ignored the correlation. In general, however, data correlation could affect results of stochastic simulation, as shown by the validation results.

INTRODUCTION

Generally, a mathematical model is used in stochastic simulation studies. In addition to randomness, correlation may exist among the variables or parameters of such models. In the case of forest ecosystems, for example, growth can be influenced by correlated variables, such as temperatures and precipitations. Similarly, in complex forest product technologies, predicted production, or returns, may depend on several engineering and economic variables, some of which may be correlated. However, in most applications of Monte Carlo simulation in forestry or forest products, data correlation has been largely ignored (e.g., Engelhard and Anderson, 1983).

Stochastic simulation is a practical approach to prediction because estimating a likelihood distribution for many variables in a model is often easier than estimating their precise values. In this paper we shall first review classical stochastic (Monte Carlo) simulation techniques, then suggest a method to take into account the subjective correlations among variables, validate the method, and apply it to a specific case study.

MONTE CARLO TECHNIQUE

The Monte Carlo simulation technique utilizes three essential elements: (1) a mathematical model to calculate a discrete numerical result or outcome as a function of one or more discrete variables, (2) a sequence of random (or pseudorandom) numbers to represent random probabilities, and (3) probability density functions or cumulative distribution functions for the random variables of the model.

The mathematical model is used repetitively to calculate a large sample of outcomes from different values assigned to the random variables. The sequence of random numbers is generally drawn independently from the uniform distribution on the unit interval (0,1) and thus represents a sequence of so-called uniform deviates. The "distribution" of a continuous random variable refers to the probability of its occurrence over its domain or "distribution space."

The distribution of a random variable is represented mathematically by its probability density function, which gives the probability that the random variable will occur within any subspace of the distribution space. Examples of probability

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[1] For presentation at Symposium on Systems Analysis in Forest Resources, Charleston, South Carolina, March 3-7, 1991. Research supported by the USDA Forest Service, Forest Products Laboratory, and by the School of Natural Resources, University of Wisconsin, Madison The Forest Products Laboratory is maintained in cooperation with the University of Wisconsin. This article was written and prepared by U.S. Government employees on official time, and it is therefore in the public domain and not subject to copyright.

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density functions include the uniform, triangular, exponential, gamma, beta, Weibull, lognormal, and normal distributions (e.g., see Fishman, 1973, pp. 200-214). Probability density functions for multivariate distributions cover the case in which two or more random variables are correlated with one another, including for example the multivariate normal distribution (Hogg and Craig, 1978, p. 409).

The distribution function (or cumulative distribution function) for a continuous random variable is the probability that the random variable will be less than or equal to a given value, *x*, within the distribution space (Hogg and Craig, 1978, p. 31). The cumulative distribution function is the integral of the probability density function over the interval from negative infinity to *x*.

Because all probabilities range from 0 to 1, the cumulative distribution function is bounded by the unit interval (0, 1). Based on this observation, one can transform a sequence of independent uniform deviates into a sample of pseudorandom variates that correspond to a specified probability distribution.

For example, the method of Box and Muller (1958) transforms independent uniform deviates U into standard normal variates z of mean zero and variance 1 by the formula

$$z = \left(-2\log U\right)^{1/2} \cos 2\pi U \qquad (1)$$

In addition, techniques are available for transforming standard normal variates into random variates that correspond to a specified multivariate normal distribution; i.e., with a specified vector of mean values and specified variance-covariance structure (Scheuer and Stoller, 1962).

In summary, in Monte Carlo simulation, a pseudorandom number generator is used to produce a sequence of independent uniform deviates, the uniform deviates are transformed into random variables or "sample data" for random variables in a mathematical model, and the "sample data" are then used in the model to compute a corresponding "sample" of outcomes from which the simulated outcome distribution can be inferred.

SIMULATIONS BASED ON MULTIVARIATE NORMAL DISTRIBUTION

Few studies deal with multivariate stochastic simulation with correlated variables. Such studies include methodological reports (e.g., Oren, 1981; Shachter and Kenley, 1989) and some applications, including those by Kaya and Buongiorno (1989), Bianchi et al. (1978), Van der Knoop and Hooijmans (1989), and Schwert (1989). There appear to be few published examples of multivariate Monte Carlo simulation techniques in risk analysis studies or decision theory (Hertz and Thomas, 1983 and 1984; Merkhofer, 1987) or in the areas of forestry or forest products. Nevertheless, techniques have been described in the literature that consider covariance or joint probability distributions among random normal variables in Monte Carlo simulation. The introduction of correlation significantly increases the complexity of generating random variates. Instead of generating each variate independently as in the conventional simulation, sample vectors of pseudorandom data must be generated such that all the variates exhibit the appropriate covariance structure.

An early multivariate simulation technique, based on the multivariate normal distribution, was developed by Scheuer and Stoller of the Rand Corporation (Scheuer and Stoller, 1962). The Scheuer and Stoller algorithm presupposes estimates of (1) the covariance matrix Σ containing the variance and covariances of the normally distributed random variables (e.g., see Hogg and Craig, 1978, pp. 408-409) and (2) the vector of their means μ. The algorithm is based on the theorem (Anderson, 1958, p. 19) that if x is an n-component vector of variables with a multivariate normal distribution (i.e., x is distributed as N(μ, Σ), where μ is the vector of mean values and Σ is the covariance matrix), then a vector of variates x that correspond to the multivariate normal distribution of x is given by

$$\mathbf{x} = \mathbf{C}\mathbf{z} + \mu \qquad (2)$$

where **C** is the Cholesky Factor of Σ and **z** is an ***n***-component vector of independent standard normal variates. The matrix **C** is the unique lower-triangular matrix such that

$$\mathbf{C}\mathbf{C}' = \Sigma \qquad (3)$$

Thus, finding **C** (Cholesky factorization) is equivalent to finding the "square root" of the covariance matrix.

For the purpose of Monte Carlo simulation, a pseudorandom vector **z** can be derived by the method of Box and Muller from a sequence of uniform deviates. as from Equation (1), such that **z** corresponds to independent pseudorandom variates drawn from the standard normal distribution **N(0, I),** having a mean vector of **0** and covariance matrix equal to the identity matrix **I.**

Thus, once the Cholesky matrix C has been derived from the covariance matrix Σ, repeated application of Equation (2) to successive random vectors **z** yields a large sample of independent pseudorandom data vectors, each vector corresponding to an independent observation of the variables from the specified multivariate normal distribution. Computer programs are available to obtain the Cholesky Factor of a matrix, such as the programs in IMSL MATH/LIBRARY (IMSL, Inc., 1987).[4]

Requirement of Positive Definiteness

The algorithm of Scheuer and Stoller requires that the matrix **C** in Equation (3) be computable. Previous authors have noted that this Cholesky factorization or triangular decomposition of the symmetric matrix Σ is possible if and only if Σ is positive definite (see Farebrother and Berry, 1974). This may not be the case for a covariance matrix that has been established by judgement rather than from real data. Consider, for example, the covariance matrix

$$\begin{vmatrix} 1.0 & 0.99 & 0.99 \\ 0.99 & 1.0 & -0.99 \\ 0.99 & -0.99 & 1.0 \end{vmatrix}$$

This matrix is symmetric, and it might seem to be an acceptable covariance matrix for a three-dimensional set of variable-s (xl, x_2, and x_3). The matrix would indicate that x_1 is very highly correlated with x_2 and x_3, while x_2 is strongly negatively correlated with x_3. However, if the vector **a** is defined as (-1, 1, 1) and the variable y is defined as y = **a'x** (a linear combination of random variables, itself a random variable), then var(y) = **a'Σa.** But this leads to a negative variance: var(y) = -2.94, and thus the matrix Σ cannot be a representation of the covariance structure of **x.** For the quadratic form **a'Σa** to be positive, for any non-negative **a,** Σ would have to be positive definite, by definition (Searle, 1971, p. 34).

This requirement of positive definiteness creates a problem for multivariate stochastic simulation whenever distributional parameters are obtained subjectively (e.g., from expert opinion or from various sources via statistical inference) because it is possible to propose a subjective "covariance"

[4] The use of trade or firm names in this publication is for reader information and does not imply endorsement by the U.S. Department of Agriculture of any product or service.

matrix that seems reasonable but that is actually not a true covariance matrix because it is not positive definite. As the number of correlated variables increases, the problem tends to become more severe. At the same time, the requirement of positive definiteness provides a natural test of the proposed covariance matrix. Once positive definiteness is satisfied, one is confident that the covariance matrix used in the simulation does indeed represent a relationship that would be possible among real data.

Subjective Definition of Multivariate Normal Distributions

As we have indicated, the key problem in doing multivariate stochastic simulation with judgmental data is to define a covariance matrix Σ that is positive definite. Oren (1981) provided one method (conjugate directions) for solving the problem. Oren's method was based on construction of a matrix of the means and conditional means of the variables of interest. In the process, he suggested a way of ensuring that the covariance matrix of the variables would be positive definite.

Though insightful, Oren's method seems difficult to apply because of the lack of a practical method to estimate all the conditional expectations needed. Shachter and Kenley (1989) did offer a systematic approach to estimating conditionality parameters (the method of the "Gaussian influence diagram") that could then be used to derive a covariance matrix, but the technique seems complicated and requires the estimation of parameters that cannot be explained easily to non-statisticians.

Nevertheless, one aspect of Oren's approach was extremely useful from a practical standpoint, when combined with the Scheuer and Stoller technique. Oren's method of restoring positive definiteness to a "covariance matrix" (by increasing the variance of certain parameters while keeping the conditionality or covariance structure constant) provided the basis for the modified Scheuer and Stoller technique.

Modified Scheuer and Stoller Technique

The Modified Scheuer and Stoller technique begins with estimates of the familiar partial correlation coefficient statistics. The complete matrix of correlation coefficients for n random variables is

$$\mathbf{P} = \begin{bmatrix} 1 & p_{12} & p_{13} & \cdots & p_{1n} \\ p_{21} & 1 & p_{23} & \cdots & p_{2n} \\ \vdots & & & \ddots & \\ p_{n1} & & \cdots & & 1 \end{bmatrix} \quad (4)$$

where p_{ij} are the partial correlation coefficients between random variables x_i and x_j

(5)

where σ_{ij} is the covariance and σ_i and σ_j are the standard deviations (Hogg and Craig, 1978, p. 73).

In practice, the meaning of a partial correlation coefficient is easy to communicate because it varies between - 1 and + 1 depending on the degree of negative or positive correlation between two variables. Also, the meaning of different values of the correlation coefficient can be readily communicated with scatterplot diagrams. Thus, subjective estimates of partial correlation coefficients are reasonably easy to obtain.

The only other data required by the modified technique are estimates of the means and standard deviations of the variables of interest. In our method, these are derived simultaneously from estimates of the 99-percent confidence interval for the expected value of each random variable.

Given the normal distribution assumption, the midpoint of the 99-percent confidence interval provides an estimate of the unconditional mean value of each variable (μ_i). In addition, the width of the 99-percent confidence interval for each variable is about 5.15 standard deviations. Thus, an estimate of the standard deviation of each variable is

$$\sigma_i = \left(x_{i\text{high}} - x_{i\text{low}}\right) / 5.15 \quad (6)$$

where $x_{i\text{high}}$ and $x_{i\text{low}}$ refer to the upper and lower bounds of the 99-percent confidence interval for the variable x_i.

Given the estimated **P** matrix and the estimated standard deviations, an initial or "preliminaty" estimate of the covariance matrix is given by the following equation (Morrison, 1967, p. 80):

$$\Sigma = \mathbf{D}(\sigma_i)\ \mathbf{P}\ \mathbf{D}(\sigma_i) \quad (7)$$

where the matrix $\mathbf{D}(\sigma_i)$ is the diagonal matrix of estimated standard deviations of the variables. Still, there is no assurance at this point that Σ is a positive definite matrix, corresponding truly to a possible covariance structure. Nevertheless, following Oren's technique for restoring positive definiteness, the Scheuer and Stoller algorithm (or Cholesky factorization) itself provides the means by which a matrix Σ can be tested for positive definiteness and can be modified, if necessary, to achieve positive definiteness. The method is based on the following observations.

First, a symmetric matrix is positive definite if and only if it can be written as **BB'**, for a nonsingular matrix **B** (see Searle, 1971, p. 36, Lemma 4 and proof). Since the Cholesky factor is defined as the unique lower-triangular matrix **C** such that **CC'** = Σ (Equation 3). then Σ is positive definite if and only if the Cholesky factor **C** is nonsingular. In addition, a square matrix such as **C** is nonsingular if and only if the determinant of the matrix is not equal to zero (e.g., see Isaak and Manougian, 1976. p. 186). Also, the determinant of a triangular matrix such as **C** is equal to the product of all the entries on its main diagonal (e.g., see Isaak and Manougian, 1976, p. 181). Thus, the lower-triangular matrix **C** (the Cholesky factor) will be nonsingular if and only if the product of all the entries on its main diagonal (the c_{ii} elements) are not equal to zero.

Furthermore, the c_{ii} elements of the Cholesky factor **C** are computed by the Scheuer and Stoller algorithm (Figure 1). Formula la gives the c_{i1} element, and formula 2a gives the c_{ii} elements for $i > 1$. The requirement that the product of the c_{ii} elements be nonzero (for **C** to be nonsingular) is equivalent to each of the c_{ii} elements being positive. This occurs if and only if the term under the square root in formula 2a (Figure 1) is a positive real number. In summary, the matrix Σ is positive definite if and only if the term under the square root sign in formula 2a for each of the c_{ii} elements of the Cholesky factor of Σ is a positive real number.

Finally, it can be observed that whenever the term under the square root sign in formula 2a is negative or zero, then increasing the corresponding variance factor (σ_i^2) by

$$\Delta_i = \sum_{k=1}^{i-1} c_{ik}^2 - \sigma_i^2 + \varepsilon_i \quad (8)$$

where ε_i is an arbitrarily small number, can restore positive definiteness to the covariance matrix Σ. Thus, the Scheuer and Stoller algorithm provides a simple way of testing if the matrix Σ is positive definite and of restoring positive definiteness if needed.

As shown by Equation (5). the effect of increasing the variance of one variable while keeping the covariance fixed is to cause any related correlation coefficients to decrease.

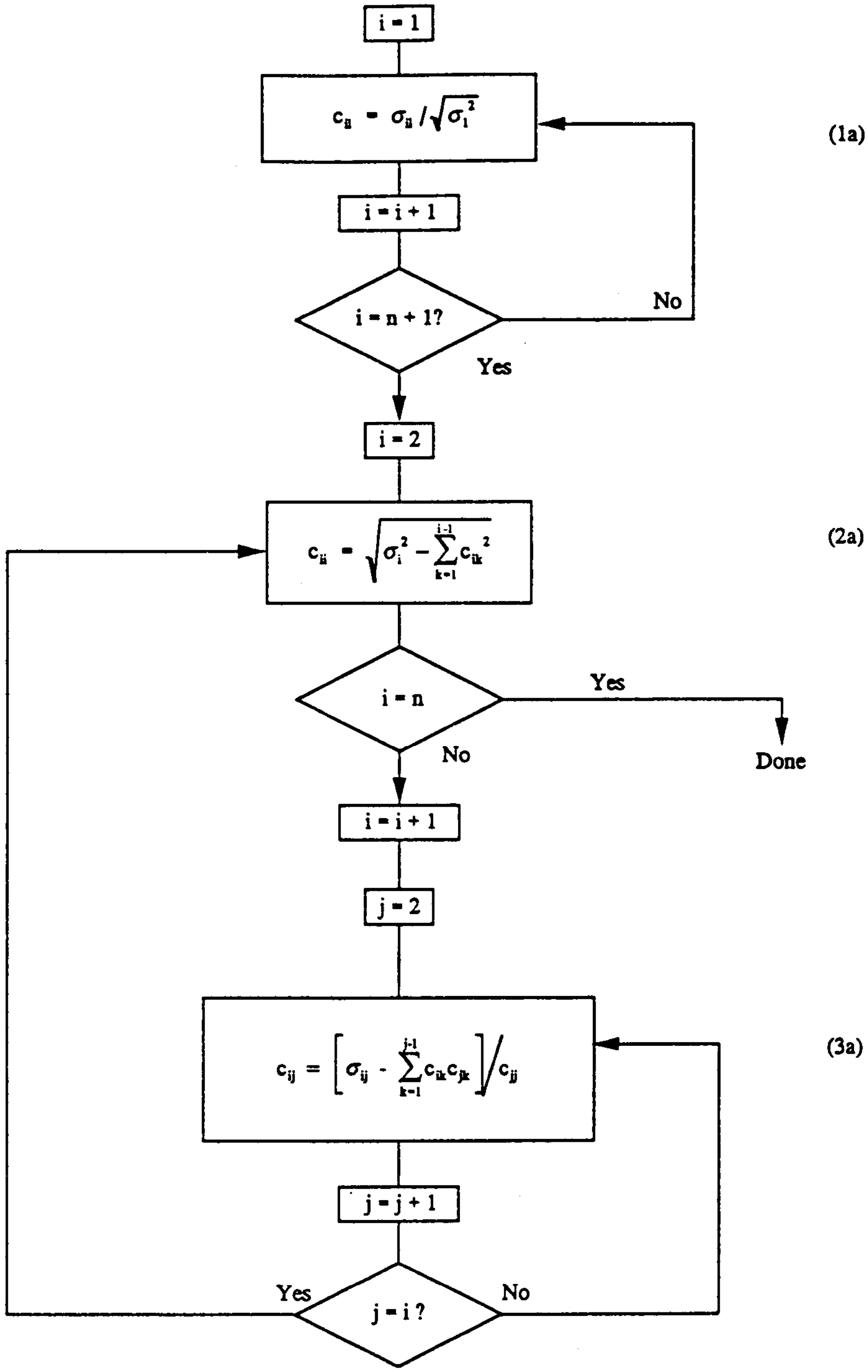

Figure 1. Scheuer and Stoller algorithm for deriving elements c_{ij} in the Cholesky factor **C** of the matrix Σ with elements σ_{ij}.

Essentially, when the initial variance and correlation structure is not acceptable because the implicit "covariance matrix" is not positive definite, the technique finds an adjusted variance and correlation structure that meets the positive definiteness requirement, while preserving the covariance terms. This may also be interpreted as a situation in which the originally specified variance of one or more variables is "too small" in relation to the larger specified variances of other variables and the specified covariance structure (see Oren, 1981, p. 34). If the revised variance and correlation structure is not acceptable-i.e., if the revised variances are judged to be too large-or if the revised correlation coefficients are too small, the entire process can be repeated, starting with revised initial estimates of the correlation coefficients and confidence intervals for the variables; i.e., with smaller initial correlation coefficient assumptions or larger initial variance assumptions, or both, resulting in different covariance terms.

Clearly, the order in which variables are entered in the matrix **P** can affect which variance is adjusted and the extent of adjustment required, However, in practice, changes in the order of the variables seem to have little impact on results of simulation. In principle, one could refine the technique further to find the order of variables in the **P** matrix that minimizes the required adjustments in variance parameters to restore positive definiteness.

Summary of Method

The Monte Carlo simulation technique with subjective specification of a multivariate normal distribution consists of the following steps:

1. Obtain subjective estimates of 99-percent confidence intervals of unconditional expected values for each random variable in the simulation model. (This is approximately equivalent to asking for the lowest and highest possible values that each random variable may take.) Use this 99-percent confidence interval to derive the estimated unconditional mean, variance, and standard deviation of each random variable (Equation 6). Arrange the mean values in a vector μ and the standard deviations on the diagonal of a diagonal matrix $D(\sigma_i)$.
2. Obtain subjective estimates of the partial correlation coefficients between each pair of random variables. The elicitation of these subjective estimates from experts can be facilitated by using scatterplot diagrams to illustrate various values of positive and negative correlation coefficients. Arrange the partial correlation coefficients in the correlation matrix **P.**
3. Derive a preliminary estimate of the covariance matrix Σ from the correlation matrix and the diagonal matrix of standard deviations (Equation 7).
4. Use the modified Scheuer and Stoller algorithm to derive the Cholesky factor **C** of the matrix Σ. The modified algorithm will stop if the term under the square root sign of formula 2a (Figure 1) is found to be negative on any iteration. In that case, the corresponding variance (σ_i^2) will need to be increased by the amount given by Equation (8).
5. If any variance term has been revised in Step 4, compute the revised correlation matrix **P** from the revised covariance matrix Σ, using Equation (7). and examine the revised partial correlation coefficients in **P** and the revised variance terms in Σ to verify that they are still acceptable. If the correlation coefficients have been decreased too much or if the variance terms have been increased too much, go back to Steps 1 and 2, and start over with revised subjective estimates for all the statistics (specifically, widen some confidence interval assumptions or decrease the absolute magnitude of some correlation coefficients, or both, and repeat the subsequent steps).
6. Use a pseudorandom number generator and Equation (1) to generate a sequence of independent standard normal variates, and arrange the sequence in *n*-dimensional vectors, denoted **z.**
7. For each **z** vector, calculate a corresponding multivariate random vector x, using the vector of estimated means μ, the Cholesky matrix **C** derived from Step 5. and Equation (2).
8. Repeat Steps 6 and 7 to generate a large number (e.g., 11,000) of independent vectors of multivariate normal variates.
9. Use the sample data generated in Step a to calculate a large sample of results predicted by the model, and examine the distribution of these results. The sample distribution can be used to infer the range and distribution of outcomes and the probability that various subsets of outcomes will occur.

An interactive microcomputer program, STOCSM, was written in FORTRAN by Ince to facilitate Steps 3 through 8 (Ince, 1990). Very recently, commercial software to do risk analysis with subjective multivariate distributions has become available (Market Engineering Co., 1991). The software, called Crystal Ball, uses rank correlation to correlate variables. This loses some information, but it permits the use of other marginal distributions besides the normal distribution. Like STOCSM, Crystal Ball starts from estimates of the marginal distribution of variables, and of the correlations between variables. Crystal Ball also revises the correlation matrix, when marginal distributions and correlations are inconsistent.

VALIDATION OF METHOD

To test the performance of the method, results of formulas for the expected values of products and quotients of pairs of correlated random variables were compared with simulated results obtained from the STOCSM program. These results also serve to illustrate how correlation assumptions may affect results of stochastic simulation.

The expected value of the product of two correlated continuous random variables *X* and *Y* is

$$E(XY) = \mu_X \mu_Y + \sigma_{XY} \quad (9)$$

The expected value of the quotient of two random variables is

$$E(X/Y) \approx \frac{\mu_X}{\mu_Y} - \frac{1}{\mu_Y^2}\sigma_{XY} + \frac{\mu_X}{\mu_Y^3}\sigma_Y^2 \quad (10)$$

(See Mood and Graybill, 1963, pp. 180-181.) Two pairs of random variables (X_1, Y_1) and (X_2, Y_2) were used in the tests. All variables had 99-percent confidence bounds of 10 and 20 and were normally distributed. Thus, they all had means of 15 and variances of 3.767 (Equation 6). X_1 and Y_1 had a correlation of -0.9, whereas X_2 and Y_2 had a correlation of +0.9. From these variances and correlations, the covariance could be obtained by Equation (5), and Equations (9) and (10) could be used to compute, analytically, the expected values of products and quotients of (X_1, Y_1) and (X_2, Y_2).

The results appear in Tables 1 and 2. The tables show that the expected value of the product of positively correlated variables (228.39) is about 3 percent higher than that of the negatively correlated variables (221.61). Instead, the expected value of the quotient of the positively correlated variables (1.002) is 3 percent lower than that of the negatively correlated variable-s (1.032).

The results in Tables 1 and 2 also show that the expected values computed by the STOCSM simulator in 1,000 replications were very close to those predicted by the analytical formulas. This supports the validity of the method embodied in STOCSM, at least for these admittedly simple cases.

APPLICATION OF METHOD

In a recently completed study at the Forest Products Laboratory (FPL), we applied the multivariate Monte Carlo technique to simulate the economic performance of a hypothetical paperboard mill with a new pulping and papermaking technology (Ince, 1990). The simulation was designed to assess the probability of investing in a 500 t/day mill producing linerboard with a new technique called CTMP-press-drying. This technique resulted from research at FPL, and a mathematical model of a CTMP-press-drying linerboard mill was developed (Ince, 1990). However, because the technology had yet to be applied commercially and because there was general uncertainty about the values that should be assigned to many technical and economic variables, eighteen variables were treated as random variables. Furthermore, we recognized that ten of those variables would be correlated and that the assumption of normality would be generally applicable. The nature of the distribution function and the magnitude of the uncertainties (standard deviations and correlations) were

Table 1: **Comparison of STOCSM results with those obtained from analytical formulas: Product of variables.**

Random variable	Mean	Variance	Correlation	Expected Value of XY Analytical	Expected Value of XY Simulated
X_1	15	3.767	-0.9	221.61	221.46
Y_1	15	3.767			
X_2	15	3.767	+0.9	228.39	227.92
Y_2	15	3.767			

Table 2: **Comparison of STOCSM results with those obtained from analytical formulas: Quotient of variables.**

Random variable	Mean	Variance	Correlation	Expected Value of X/Y Analytical	Expected Value of X/Y Simulated
X_1	15	3.767	-0.9	1.032	1.033
Y_1	15	3.767			
X_2	15	3.767	+0.9	1.002	1.001
Y_2	15	3.767			

obtained by interviewing scientists, engineers, and economists at the FPL (Ince, 1990). Figure 2 shows the correlation matrix that was obtained for the ten correlated variables in the model.

The STOCSM program was then used to predict the distribution of the internal rate of return (IRR) and net present value of the mill, given this subjective assessment of the distribution of the stochastic variables. Two simulations were done. In one simulation, the full multivariate normal distribution was considered, including the correlation coefficients shown in Figure 2. In the other simulation, variable correlations were ignored; thus, all the variables were regarded as random but independently distributed.

The results of the simulation experiments, with 1,000 observations each, appear in Table 3. The IRR for the hypothetical CTMP-press-drying paperboard mill is expressed in real dollars, after taxes. The most striking aspect of the results is that they differ so little. The means of the IRR were almost identical in the two cases, although the standard deviation was 8 percent smaller when the correlation of variables was taken into account. Thus, although ignoring the correlations would lead to exaggerating the risk of the investment, the difference between the results obtained by the simulations is quite small. Still, one should not make broad generalizations from this example. In other cases, it is conceivable that larger differences could occur between multivariate stochastic simulation and conventional Monte Carlo simulation, depending on the distributional assumptions. This was in part demonstrated earlier by the experiments with the products and quotients of highly correlated variables (Tables 1 and 2).

CONCLUSIONS

Generally applicable techniques for subjective assessment of a multivariate normal distribution were developed and demonstrated in the context of Monte Carlo simulation. The techniques were based on subjective specification of the 99-percent confidence intervals and partial correlation coefficients for the random variables. The technique combines the modified Scheuer and Stoller algorithm for Cholesky factorization and the variance-adjustment technique suggested by Oren, for ensuring a positive definite covariance matrix. The technique provides the means for assessing a subjective multivariate normal distribution-starting from the 99-percent confidence intervals and partial correlation coefficients, checking the covariance matrix for positive definiteness, and restoring positive definiteness, if necessary, by adjusting the variance assumptions while preserving the covariance terms. The method was validated by showing that simulated expected values of products and quotients of highly correlated variables were very close to those obtained from analytical formulas.

The method was applied to predict the internal rate of return of an investment in a new papermaking facility, with ten

		1	2	3	4	5	6	7	8	9
		.4	.7							
1	.4							.3	.3	.8
2	.7									
3					-.7	-.5	-.7			
4				-.7		-.5	.7			
5				-.5	-.5					
6				-.7	.7					
7		.3							.5	.5
8		.3						.5		.5
9		.8						.5	.5	

1.Specific gravity of pulpwood
2.Price of pulpwood ($,1986/cord)
3.Chipping energy requirement (kWh/ton)
4.Pulp yield (percent)
5.Chemical charge in chip pretreatment (lb/ton)
6.Electric energy in refining (kWh/ton of chips)
7.Cost of pulping effluent facilities ($,1986)
8.Price of electricity ($,1986/MCF)
9.Price of linearboard ($,1986/ton)

Figure 2. Matrix of correlation coefficients for ten correlated random variables in simulation experiment.

Table 3: Mean and standard deviation of internal rate of return for hypothetical CTMP-press-drying paperboard mill.

	Internal rate of return	
Monte Carlo simulation	Mean	Standard deviation
Multivariate normal distribution	8.55	1.28
Conventional	8.53	1.39

correlated random variables. The results showed that recognizing the correlation among variables had no effect on the expected internal rate of return, although it decreased its variance. The difference was small, in this particular case. Additional experiments are needed to determine under what circumstances correlations among variables in Monte Carlo simulations should be recognized explicitly.

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