

LATERAL STABILITY OF BEAMS WITH ELASTIC END RESTRAINTS

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ABSTRACT: In the analysis of the lateral buckling of simply supported beams, the ends are assumed to be rigidly restrained against tip. Real supports are, of course, never perfectly rigid. This report examines the relation of the stiffness of the end axial rotation restraint to the buckling load when the stabilizing effect of an attached deck is taken into account. For the case of uniform load, it is found that as the restraint stiffness approaches zero, the buckling load also approaches zero. This has implications in the design of large roof systems where end restraint on one member is provided by the torsional rigidity of another member connected in-line. Families of design curves are presented which show the effects of restraint stiffness, span-depth ratio, and shear stiffness of attached roof deck. It is concluded that periodic bracing against axial rotation is essential for stability of long roof systems with several beams spliced together in-line.

INTRODUCTION

This paper analyzes the effect of elastic end axial-rotation (tip) restraint upon the buckling load of simply supported beams with shear stiffness of an attached deck taken into account. The deck is assumed to be "hinged" to the top edge of the beam, so that only its in-plane shear stiffness resists buckling.

In the design of large flat roof systems it is quite common for one beam of the system to be simply supported by adjacent beams connected in-line. The center member in Fig. 1 is an example. In analyzing the lateral stability of the center member the question arises as to whether its ends are prevented from tipping as was assumed in deriving the formula for the lateral buckling load of a "simply supported" beam. Obviously the tip restraint at a shear connection is elastic rather than rigid. If the only rigid tip restraint in the system in Fig. 1 is provided at the outer walls then the torsional rigidity of the outer beams determines the elasticity of the tip restraint (rotational stiffness of the support) at the shear connection. This restraint stiffness could in some cases be rather small.

Flint (1) recommended a simple linear reduction factor on the buckling load when the ends are only elastically restrained against axial rotation. However, his formula is only accurate for very stiff restraints and thus not suited to the analysis of long roof systems considered here. Trahair (3) treated the title problem in a more general context of various end conditions. However, his analysis did not include the stabilizing effect of an attached deck which is a major consideration in many roof systems, especially wood roofs.

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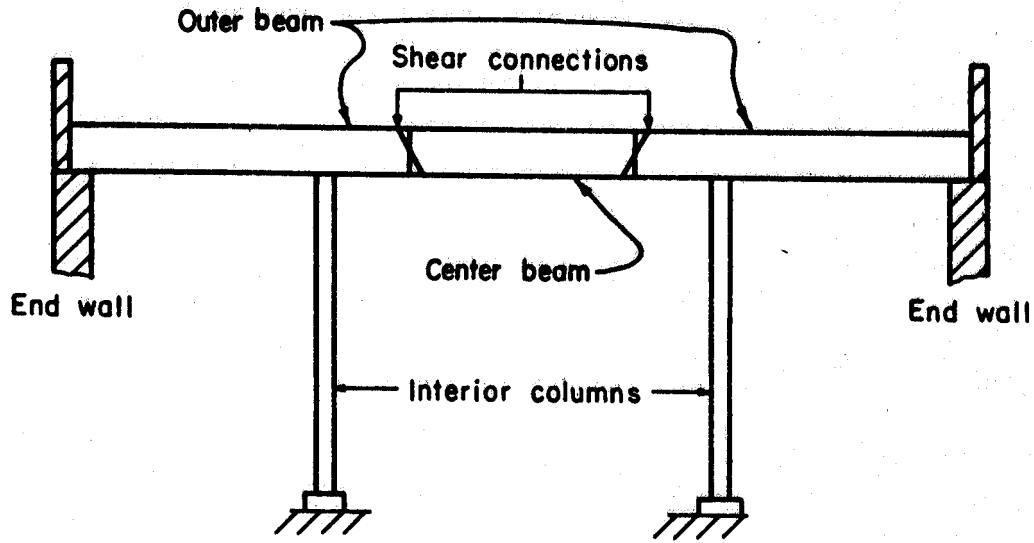


FIG. 1.—Flat Roof System With Rigid Tip Restraints at End Walls. Center Beam is Restrained Against Tip by Shear Connections to Outer Beams (M152046)

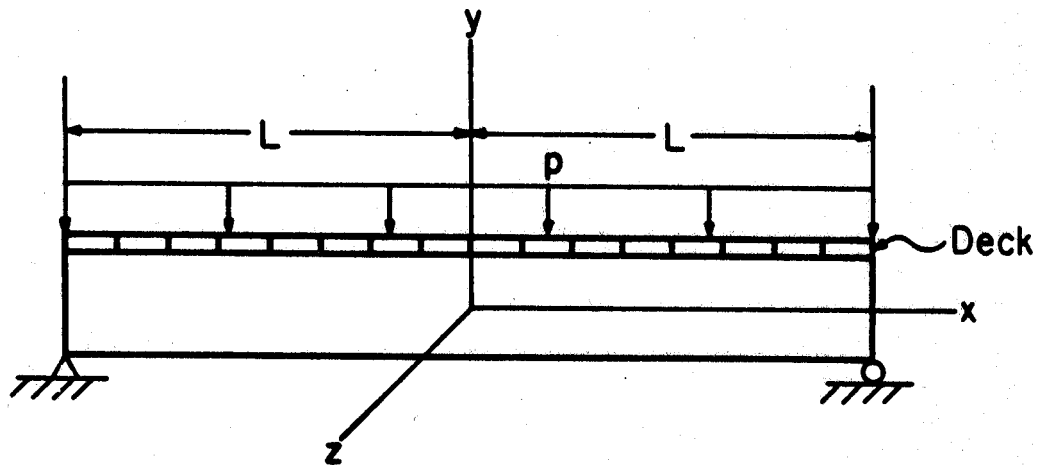


FIG. 2.—Simply Supported Beam-and-Deck System. Ends are Elastically Restrained Against Tip (M145355)

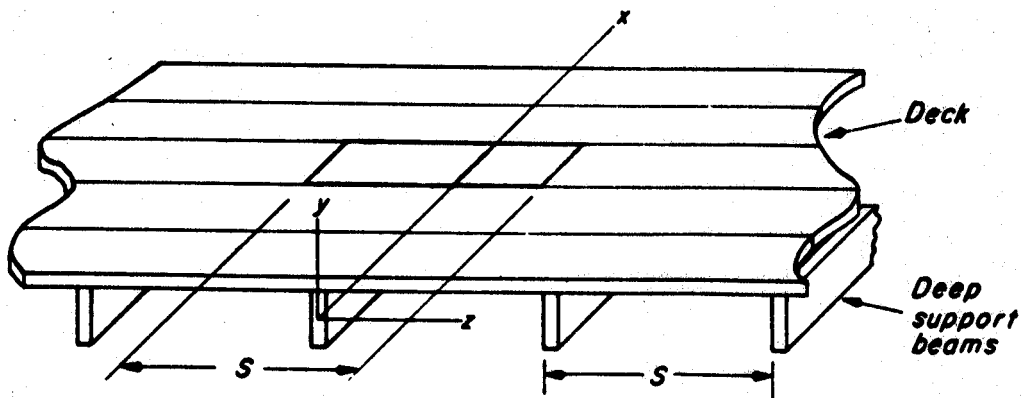


FIG. 3.—Sketch of Beam-and-Deck System (M152046)

DIFFERENTIAL EQUATIONS AND BOUNDARY CONDITIONS

The system to be analyzed is shown in Fig. 2. By symmetry we need to treat only the right half. Following earlier work (4,5), the total energy of the system is

$$U = \frac{1}{2} \int_0^L \left\{ EI_y \left(\frac{d^2 w}{dx^2} \right)^2 + GK \left(\frac{d\beta}{dx} \right)^2 - 2 \frac{d(w_i + w)}{dx} \frac{d[M(\beta_i + \beta)]}{dx} + SG_D \left[\frac{d}{dx} (w + c\beta) \right]^2 - pc[(\beta_i + \beta)^2 - \beta_i^2] \right\} dx + \frac{1}{2} R\beta_L^2 \dots\dots\dots (1)$$

in which EI_y = lateral flexural rigidity; GK = torsional rigidity; w = lateral displacement of neutral axis due to load; β = angle of twist due to load; w_i , β_i = initial deformations; β_L = rotation of elastic tip restraint at ends; M = internal bending moment about z -axis; S = spacing between beams in roof system (i.e., width of deck associated with one beam), see Fig. 3; G_D = in-plane shear rigidity of deck (force per unit length of edge); c = distance from shear center to top edge of beam; p = downward distributed load (force per unit length); L = half-length of beam; and R = stiffness of elastic tip restraint (torque per angle). This expression follows Refs. 4 and 5 except that the last term is added here to account for the energy stored in the elastic tip restraints. A condition of equilibrium may be invoked by requiring the first variation of U to vanish. This step yields the following:

Differential Equations

$$EI_y \frac{d^2 w}{dx^2} = -M(\beta_i + \beta) + SG_D(w + c\beta) + C_1 x + C_2 \dots\dots\dots (2)$$

$$(GK + SG_D c^2) \frac{d^2 \beta}{dx^2} = M \frac{d^2(w_i + w)}{dx^2} - SG_D c \frac{d^2 w}{dx^2} - pc(\beta_i + \beta) \dots\dots\dots (3)$$

in which C_1 and C_2 are constants of integration.

Boundary Conditions at $x = 0$

$$C_1 \delta w = 0 \dots\dots\dots (4)$$

$$\frac{d^2 w}{dx^2} \cdot \delta \frac{dw}{dx} = 0 \dots\dots\dots (5)$$

$$\left\{ -GK \frac{d\beta}{dx} - SG_D c \left(\frac{dw}{dx} + c \frac{d\beta}{dx} \right) + M \frac{d}{dx} (w_i + w) \right\} \delta \beta = 0 \dots\dots\dots (6)$$

Boundary Conditions at $x = L$

$$C_1 \delta w = 0 \dots\dots\dots (7)$$

$$\frac{d^2 w}{dx^2} \cdot \delta \frac{dw}{dx} = 0 \dots\dots\dots (8)$$

$$\left\{ GK \frac{d\beta}{dx} + SG_{Dc} \left(\frac{dw}{dx} + c \frac{d\beta}{dx} \right) - M \frac{d}{dx} (w_i + w) + R\beta \right\} \delta\beta = 0 \dots\dots\dots (9)$$

in which \mathbf{d} denotes the first variation.

APPLICATION TO SPECIFIC CASES

We shall consider three cases, all simply supported and with elastic torsion restraints at the ends: (1) Constant bending moment—zero initial deformations; (2) constant bending moment—nonzero initial deformations; (3) uniform load—zero initial deformations. We shall see that the elasticity of the tip restraints has no effect on the buckling load in Cases 1 and 2, but reduces the buckling load in Case 3.

Case 1. Constant Bending Moment—Zero Initial Deformations.—In this case the differential Eqs. 2 and 3 become

$$EI_y \frac{d^2w}{dx^2} = -(M - SG_{Dc}) \beta + SG_D w + C_1 x + C_2 \dots\dots\dots (10)$$

$$(GK + SG_{Dc}^2) \frac{d^2\beta}{dx^2} = (M - SG_{Dc}) \frac{d^2w}{dx^2} \dots\dots\dots (11)$$

where M is the constant bending moment, and the boundary conditions, Eqs. 4–9, become

$$C_1 = 0 \dots\dots\dots (12)$$

$$\frac{dw}{dx} = 0 \quad \text{at} \quad x = 0 \dots\dots\dots (13)$$

$$\frac{d\beta}{dx} = 0 \quad \text{at} \quad x = 0 \dots\dots\dots (14)$$

$$w = 0 \quad \text{at} \quad x = L \dots\dots\dots (15)$$

$$\frac{d^2w}{dx^2} = 0 \quad \text{at} \quad x = L \dots\dots\dots (16)$$

$$(GK + SG_{Dc}^2) \frac{d\beta}{dx} - (M - SG_{Dc}) \frac{dw}{dx} + R\beta = 0 \quad \text{at} \quad x = L \dots\dots\dots (17)$$

The solution of this boundary value problem which satisfies every condition except Eq. 16 is

$$w = A (\cos \lambda x - \cos \lambda L) \dots\dots\dots (18)$$

$$\beta = \frac{M - SG_{Dc}}{GK + SG_{Dc}^2} A \left(\cos \lambda x - 2\lambda \sin \lambda L - \frac{R}{GK + SG_{Dc}^2} \cos \lambda L \right) \dots\dots\dots (19)$$

in which A is an integration constant, and

$$\lambda^2 = \frac{(M - SG_{Dc})^2}{EI_y (GK + SG_{Dc}^2)} - \frac{SG_D}{EI_y} \dots\dots\dots (20)$$

Eq. 16 can be satisfied by setting

$$\cos \lambda L = 0, \quad \text{that is } \lambda = \frac{\pi}{2L} \dots\dots\dots (21)$$

which yields

$$M_{cr} = SG_{DC} \pm \sqrt{(GK + SG_{DC}^2) \left(\frac{\pi^2}{4L^2} EI_y + SG_D \right)} \dots\dots\dots (22)$$

the same critical buckling load as in the case where the ends are rigidity restrained against tip. This result seems unrealistic. Should not increased flexibility lower the buckling load? Perhaps the reason it does not is that the theoretical model assumes the member to be initially perfectly straight—an unrealistic assumption that may prevent the mathematical model from ever “seeing” the stiffness of the end restraints. Accordingly, we consider a case in which the beam has initial deformations.

Case 2. Constant Bending Moment—Nonzero Initial Deformations.—

The initial deformations are, for simplicity of analysis, assumed to have the same form as the eigenfunction:

$$w_i = A \cos \frac{\pi x}{2L} \dots\dots\dots (23)$$

$$\beta_i = B \cos \frac{\pi x}{2L} \dots\dots\dots (24)$$

The differential Eqs. 2 and 3 become

$$EI_y \frac{d^2 w}{dx^2} + (M - SG_{DC}) \beta - SG_D w = -MB \cos \frac{\pi x}{2L} + C_1 x + C_2 \dots\dots\dots (25)$$

$$(GK + SG_{DC}^2) \frac{d^2 \beta}{dx^2} - (M - SG_{DC}) \frac{d^2 w}{dx^2} = -MA \frac{\pi^2}{4L^2} \cos \frac{\pi x}{2L} \dots\dots\dots (26)$$

with the same boundary conditions as Case 1 except that Eq. 17 is replaced by

$$(GK + SG_{DC}^2) \frac{d\beta}{dx} - (M - SG_{DC}) \frac{dw}{dx} + MA \frac{\pi}{2L} + R\beta = 0 \quad \text{at } x = L \dots (27)$$

The solution of this boundary value problem is

$$w = \hat{A} \left[\cos \frac{\pi x}{2L} - \frac{\pi^2}{4L^2 \lambda^2} (\cos \lambda x - \cos \lambda L) \right] \dots\dots\dots (28)$$

$$\beta = \hat{B} \cos \frac{\pi x}{2L} - \frac{M - SG_{DC}}{GK + SG_{DC}^2} \hat{A} \frac{\pi^2}{4L^2 \lambda^2} (\cos \lambda x - \cos \lambda L) - \frac{\pi}{2LR} [MA + (M - SG_{DC}) \hat{A} - (GK + SG_{DC}^2) \hat{B}] \dots\dots\dots (29)$$

in which I^2 is the same as in Case 1 (see Eq. 20) and

$$\hat{A} = M \cdot \frac{(M - SG_{DC}) A + (GK + SG_{DC}^2) B}{EI_y (GK + SG_{DC}^2) \left(\frac{\pi^2}{4L^2} - \lambda^2 \right)} \dots\dots\dots (30)$$

$$\hat{B} = M \cdot \frac{\left(\frac{\pi^2}{4L^2} EI_y + SG_D\right) A + (M - SG_D c) B}{EI_y (GK + SG_D c^2) \left(\frac{\pi^2}{4L^2} - \lambda^2\right)} \dots (31)$$

Note the following:

$$(a) w, \beta \propto \frac{1}{\frac{\pi^2}{4L^2} - \lambda^2} \dots (32)$$

so that the deflections approach infinity as A approaches $p/2L$ the same eigenvalue found in Case 1.

(b) The elastically restrained ends rotate under load no matter how small the load or how large the restraint stiffness. This is seen in the fact that

$$\text{at } x = L, \beta \propto \frac{M}{K} \cdot \frac{\text{(quadratic in } M)}{\frac{\pi^2}{4L^2} - \lambda^2} \dots (33)$$

This behavior is sketched in Fig. 4 which incorporates Cases 1 and 2.

It is clear that even though the mathematical model now explicitly displays the effect of the end restraint stiffness, R , the critical buckling load is still independent of R . In fact, it is now evident that whenever the differential Eqs. 2 and 3 have constant coefficients, the critical load will

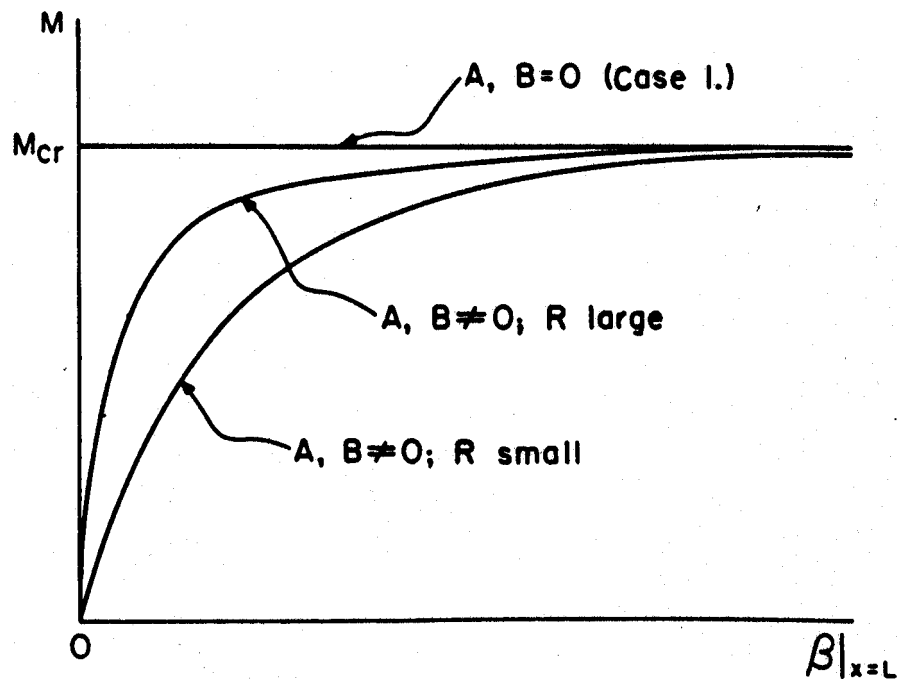


FIG. 4.—Diagrammatic Sketch of Relationship Between Load and End Rotation Given by Eq. 31. M_{cr} Corresponds to $1 = p/2L$. R is End Restraint Stiffness; A and B are Amplitudes of Initial Deformations (M152047)

depend only on those coefficients. The constraint stiffness, R , enters the problem only through the boundary condition, Eq. 9. These two cases raise the question of whether R can ever exert a strong influence on the critical load. The answer is yes, provided the differential equation has variable coefficients as it will whenever the bending moment is a function of x . The next case illustrates this.

Case 3. Uniform Load—Zero Initial Deformation.—Under a uniformly distributed load, the bending moment is

$$M = \frac{p}{2}(L^2 - x^2) \dots\dots\dots (34)$$

Introduce the nondimensional notation

$$\begin{aligned} \phi &\equiv \frac{c}{L} \sqrt{\frac{EI_y}{GK}} & \omega &\equiv \frac{w}{L} & \psi &\equiv \beta \sqrt{\frac{GK}{EI_y}} & \xi &\equiv \frac{x}{L} \\ \theta &\equiv \frac{\frac{1}{2}pL^3}{\sqrt{GK}EI_y} & \tau &\equiv \frac{SG_D L^2}{EI_y} & q &\equiv \frac{RL}{GK} \dots\dots\dots (35) \end{aligned}$$

and let a prime denote differentiation with respect to \mathbf{x} . In the absence of initial deformations, differential Eqs. 2 and 3 then become

$$\omega'' = \tau\omega - [\theta(1 - \xi^2) - \tau\phi]\psi + C_1\xi + C_2 \dots\dots\dots (36)$$

$$(1 + \tau\phi^2)\beta'' = -2\phi\theta\psi' + [\theta(1 - \xi^2) - \tau\phi]\omega'' \dots\dots\dots (37)$$

and the boundary conditions, Eqs. 4 through 9, become

$$C_1 = 0 \dots\dots\dots (38)$$

$$\omega'_0 = 0 \dots\dots\dots (39)$$

$$\psi'_0 = 0 \dots\dots\dots (40)$$

$$\omega_1 = 0 \dots\dots\dots (41)$$

$$\omega''_1 = 0 \dots\dots\dots (42)$$

$$(1 + \tau\phi^2)\psi'_1 + \tau\phi\omega'_1 + q\psi_1 = 0 \dots\dots\dots (43)$$

where subscript one on \mathbf{w} or \mathbf{y} denotes evaluation at $\mathbf{x} = 1$ and subscript zero denotes evaluation at $\mathbf{x} = 0$. Eq. 38 eliminates one integration constant. Eq. 42 shows that C_2 does not vanish. Eqs. 39 and 40 can be satisfied by taking ϕ and ψ even in \mathbf{x} . Thus, the form of solution must contain C_2 and two other integration constants, say a_0 and b_0 . We assume a power series in \mathbf{x} and \mathbf{q} :

$$\omega = a_0 \sum \sum A_{mn} \theta^m \xi^{2n} + b_0 \sum \sum B_{mn} \theta^m \xi^{2n} + C_2 \sum \sum E_{mn} \theta^m \xi^{2n} \dots\dots\dots (44)$$

$$\beta = a_0 \sum \sum C_{mn} \theta^m \xi^{2n} + b_0 \sum \sum D_{mn} \theta^m \xi^{2n} + C_2 \sum \sum F_{mn} \theta^m \xi^{2n} \dots\dots\dots (45)$$

The three boundary condition Eqs. 41, 42, and 43 then show that the integration constants a_0 , b_0 , and C_2 must all be zero unless the load parameter \mathbf{q} assumes a critical value such that the determinant of the

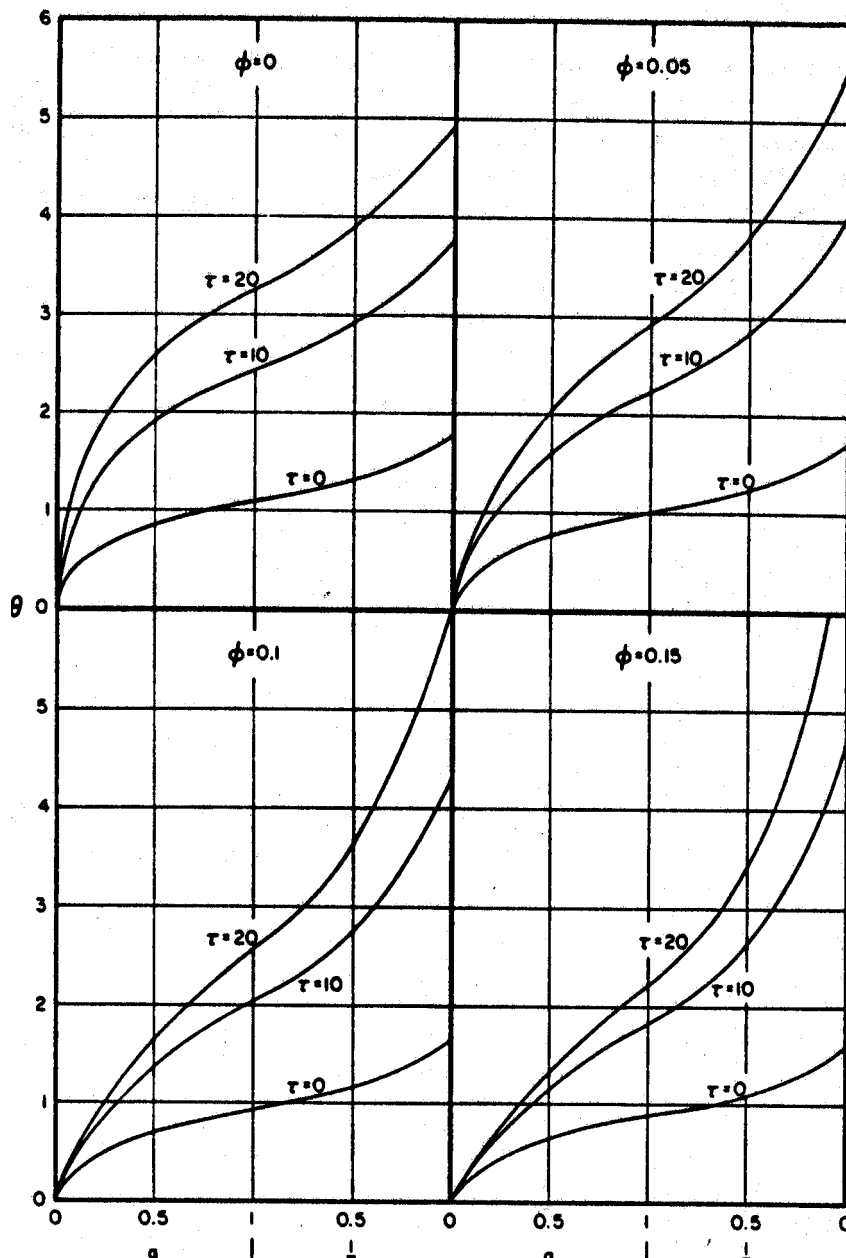


FIG. 5.—Dependence of Eigenvalue q^0 ($1/2$) $pL^3/GKEI_2$ on q^0 RL/GK . $L = 1/2$ Length, $R =$ Stiffness of End Tip Restraints. t is a Deck Stiffness Parameter (See, Eq. 33) (M145363)

coefficients of these three equations vanishes. This determinant depends on the load parameter q and its roots are the eigenvalues of the system.

A digital computer was employed to carry out the following program: (a) generate arrays A_{mn} through F_{mn} using recursion relations obtained from the differential Eqs. 36 and 37; and (b) choose q by trial such that the determinant of the coefficients of a_0 , b_0 , and C_2 in Eqs. 41, 42, and 43 vanishes.

Some results of this program are shown in Fig. 5. They agree with Ref. 4 when $q = \infty$ and the values at $t = q$ agree with those found by Trahair (3). It is seen that the buckling parameter q_{cr} depends strongly upon q , the end restraint stiffness parameter. This result has serious im-

lications for the design of systems in which a beam is simply supported by adjacent in-line beams whose torsional rigidities provide elastic tip restraints at the points of support, as will be shown by the following example.

ILLUSTRATIVE EXAMPLE

Consider the system shown in Fig. 6, in which a uniformly loaded center beam is supported by shear splices to cantilever beams in-line at each end. This example is somewhat less realistic than the system of Fig. 1, but it has the advantage of being completely analyzable by known

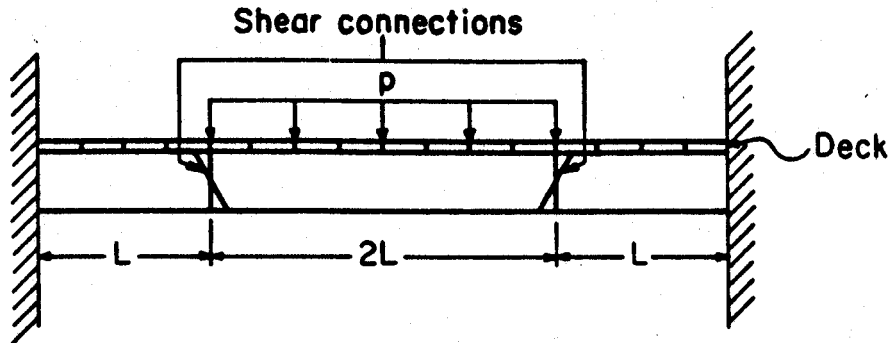


FIG. 6.—Uniformly Loaded Beam-and-Deck Supported by Cantilevers In-Line at Each End (M152048)

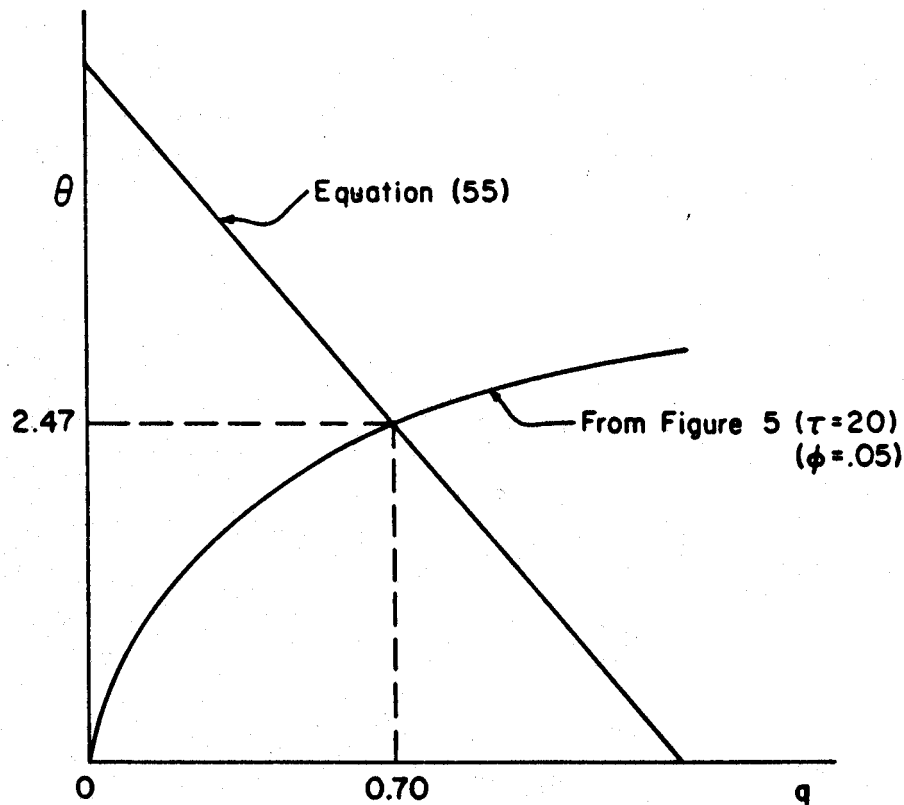


FIG. 7.—Superposition of Eq. 55 onto Fig. 5 Yields the Solution for q and θ (M152049)

results and will provide insight into other cases as well.

We shall calculate the lateral buckling load of the center member. At first we neglect the effect of the load upon the effective torsional rigidity of the outer cantilevers. Then the restraint stiffness is simply

$$R = \left(\frac{GK}{\text{length}} \right)_{\text{outer beam}} \dots\dots\dots (46)$$

and the torsion restraint parameter, q , is

$$q = R \left(\frac{\text{half length}}{GK} \right)_{\text{center beam}} = 2 \dots\dots\dots (47)$$

because the center beam is twice as long as the outer ones and all beams have the same material and cross section.

Let us assume that the deck stiffness parameter t and depth-to-span parameter f have been calculated from their definitions (Eq. 35) to be

$$\tau = 20; \phi = 0.05 \dots\dots\dots (48)$$

If the center member were treated as simply supported with ends perfectly restrained against tip, the critical value of the load parameter, q , would be (3)

$$\theta_{cr} = 5.46 \dots\dots\dots (49)$$

However, the ends are only elastically restrained against tip. Using the value $q = 2$ calculated previously, Fig. 5 of this report yields

$$\theta_{cr} = 3.83 \dots\dots\dots (50)$$

The effect of load upon the torsional rigidity of the outer cantilevers should not be neglected, however. This effect will now be considered.

The effective torsional rigidity of a beam is reduced by the presence of a primary bending load and vanishes when the primary bending load reaches the critical lateral buckling value (6). Whereas Ref. 6 shows that the actual reduction of stiffness is parabolic [i.e., proportional to $1 - (P/P_{cr})^2$] a simpler and more conservative design procedure would be to use a linear reduction factor. Then one would say

$$\left(\frac{GK}{L} \right)_{\text{effective}} = \left(\frac{GK}{L} \right) \cdot \left(1 - \frac{P}{P_{cr}} \right) \dots\dots\dots (51)$$

In this example the primary bending load on the outer cantilevers is due to an end load of

$$P = pL \dots\dots\dots (52)$$

The critical value of P (from Fig. 9 of Ref. 1) is:

$$P_{cr} = 7.6 \frac{\sqrt{GKEI_y}}{L^2} \dots\dots\dots (53)$$

Thus the torsional stiffness reduction factor in Eq. 51 is

$$\left(1 - \frac{P}{P_{cr}} \right) = \left(1 - \frac{pL}{P_{cr}} \right) = \left(1 - \frac{2\theta\sqrt{GKEI_y}}{P_{cr}L^2} \right) \text{ by definition of } q$$

$$= \left(1 - \frac{2\theta}{7.6}\right) \text{ by Eq. 53} \dots\dots\dots (54)$$

Since q is reduced by this same factor, Eq. 47 becomes

$$q = 2 \cdot \left(1 - \frac{2\theta}{7.6}\right) = 2 - 0.53\theta \dots\dots\dots (55)$$

Eq. 55 and Fig. 5 now form a pair of simultaneous conditions which determine q and θ . Fig. 7 reproduces one line from Fig. 5 and adds the straight line defined by Eq. 55. The point of intersection is at

$$\theta_{cr} = 2.47; \quad q = 0.70 \dots\dots\dots (56)$$

Note that q has been reduced from 2 to 0.70; the effect of load upon torsional rigidity is far from negligible.

Thus, the critical value of the load parameter is reduced from $q_{cr} = 5.47$ with rigid tip restraints to $q_{cr} = 2.47$ when the tip restraints are provided by supporting adjacent cantilevers in-line in this example.

REVIEW OF RESULTS

It is seen that interior members of long roof systems are far less stable than elementary theory would predict. This can be attributed to the elasticity of their end rotation restraints.

In view of this, a designer should specify bracing members at convenient locations and use the torsional rigidity of adjacent in-line beams with the reduction factor of Eq. 51 to conservatively estimate the stiffness, R , of end torsion restraints on interior beams not directly restrained by bracing. It cannot be emphasized too strongly that elementary theory cannot be relied upon to check the stability of such interior members. The preceding example shows that it could yield a buckling load that is two or three times too high, even when the effect of attached decking is taken into account, as in Ref. 4. The preceding example strongly suggests that when several beams are shear-spliced together and supported by interior columns, the designer should provide periodic restraints against axial rotation. In the absence of such restraints the beams behave like one very long—and potentially unstable—member. The logical place to locate such restraints is at the column supports where the bending moment is negative and the deck is attached to the tension edge of the beam.

APPENDIX I.—REFERENCES

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APPENDIX II.— NOTATION

The following symbols are used in this paper:

A	=	integration constant in Case 1;
A, B	=	amplitudes of initial deformations in Case 2;
a_0, b_0	=	integration constants;
\hat{A}, \hat{B}	=	see Eqs. 28 and 29;
A_{mn} through F_{mn}	=	series coefficients, see Eqs. 42 and 43;
C_1, C_2	=	integration constants;
c	=	distance from shear center to top edge of beam;
EI_y	=	lateral flexural rigidity;
G_D	=	in-plane shear rigidity of deck (force per unit length of edge);
GK	=	torsional rigidity;
L	=	length or half-length of beam;
M	=	internal bending moment about z-axis;
P	=	downward end load on cantilever (force), see Eq. 51;
p	=	downward distributed load (force per unit length);
q	=	tip restraint parameter, see Eq. 33;
R	=	stiffness of elastic tip restraint (torque per unit angle);
S	=	spacing between beams;
U	=	total potential energy;
w	=	lateral displacement of neutral axis due to load;
x, y, z	=	coordinate axes, see Fig. 2;
β	=	angle of twist due to load;
δ	=	denotes first variation;
λ	=	see Eq. 20;
$\phi, \omega, \psi, \xi, \theta, \tau$	=	nondimensional parameters defined by Eq. 35;
θ	=	load parameter;
ξ	=	nondimensional axial coordinate;
τ	=	deck shear stiffness parameter;
ϕ	=	depth-to-span parameter;
Ψ	=	re-scaled angle of twist; and
ω	=	nondimensional lateral displacement.