Loss of Torsional Stiffness Caused by Beam Loading

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ABSTRACT: A beam under load can be more easily twisted or deflected laterally than when not under load. This loss of apparent stiffness is an example of the general principle that stiffness associated with a buckling displacement diminishes to zero as the buckling load is approached. This phenomenon was first reviewed and derived for beam-columns and then derived for beam-torsion members. It was found that stiffness decays linearly for beam-columns and parabolically for beam-torsion members. The results are important to designers of large flat roofs in which one beam must provide axial rotation restraint for the next beam in-line. Simple design equations are recommended.

Introduction

The design of large interconnected systems of roof beams often involves the support of one beam by adjacent beams in-line as shown in Fig. 1. This sketch shows three beams in-line with the center beam supported by shear connections to the two outer beams. The end walls provide axial rotation restraints which prevent the beams from tipping over. The support condition for the center beam is one of simple support with elastic axial rotation restraint. The stiffness of that elastic restraint influences the lateral-torsional buckling load of the center member; the value of that stiffness is derived from the torsional rigidity of the two outer beams.

The writer is concerned, therefore, with the torsional rigidity of bending members. Associated with every bending member is its lateral-torsional buckling load. As that load is approached, the effective torsional rigidity and lateral flexural rigidity diminish to zero. This is an example of the general principle that stiffness associated with a buckling displacement decays to zero as the buckling load is approached. Southwell (2) and Timoshenko and Gere (3) described the phenomenon as it applies to columns. Chen and Atsuta (1) gave a thorough treatment of it. In the example presented in Fig. 1 the two outer beams would have their torsional rigidity reduced by the very existence of their own bending loads. This in turn will further soften the restraint available to the center member. It is suggested that the torsional rigidity and lateral flexural rigidity of a beam be reduced by the factor

$$1 - \frac{p}{p_{cr}} \qquad (1)$$

in all calculations which call for them. (Here p is load and subscript cr denotes lateral-torsional buckling.)

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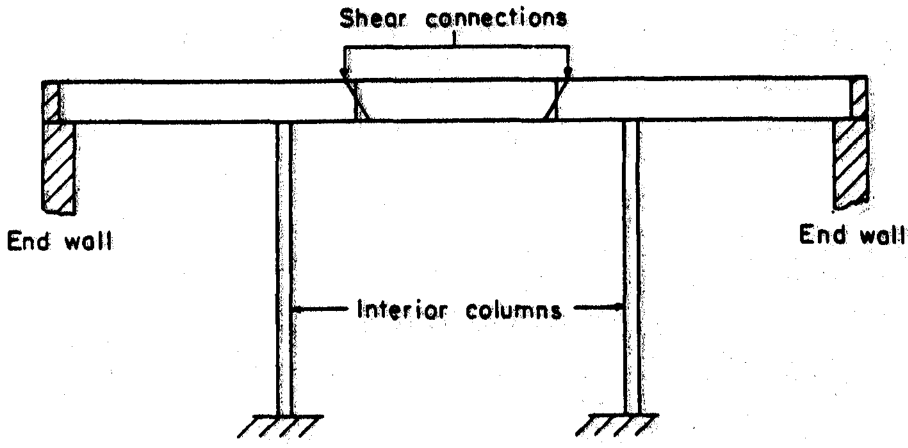

FIG 1.–Typical Flat-Roof System

Review of Beam-Column Behavior. –It is well-known that if a column has an initial lateral deflection, then that deflection will be magnified by the additon of an axial compressive force *P*. But suppose the lateral deflection is due to a lateral forace, *Q* at midspan (Fig. 2):

$$\delta = \frac{Q}{k_o} \quad \text{(2)}$$

in which d ° lateral deflection at midspan when axial force $P = 0$; and k_0 ° $48EI/L^3$ = lateral stiffness when $P = 0$. Then, the addition of an axial compressive force, *P*, will increase the midspan defection to (3):

$$\Delta = \frac{Q}{k_o} \cdot \frac{3(\tan\alpha - \alpha)}{\alpha^3} \quad \text{(3)}$$

in which $\alpha = \frac{L}{2}\sqrt{\frac{P}{EI}}$ (4)

If written $\Delta = \frac{Q}{k}$ (5)

then $k = k_o \cdot \frac{\alpha^3}{3(\tan\alpha - \alpha)}$ (6)

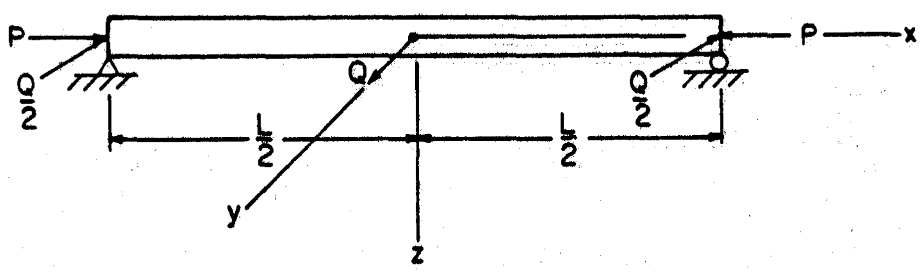

FIG 2.–Simply Supported Beam-Column

in which k = reduced lateral stiffness when P = nonzero. The reduction factor in Eq. 6 is approximately linear in *P:*

$$\frac{\alpha^3}{3(\tan\alpha - \alpha)} \approx 1 - \frac{P}{P_{cr}} \quad \text{(7)}$$

in which $P_{cr} \equiv \dfrac{\pi^2 EI}{L^2} =$ critical axial load .... (8)

Thus $k \approx k_o \left(1 - \dfrac{P}{P_{cr}}\right)$ .... (9)

This simple result will be derived next by a direct method, and the method later will be applied to the stiffness loss of beams under bending loads.

Approximate Solution via the Principle of Superposition.–Although the load-deflection relationship of this problem is nonlinear, the differential equation and boundary conditions are linear and superposition can be employed at that level. The differential equation for the beam-column shown in Fig. 2 is:

$$EIw_T'' + Pw_T = -\frac{Qx}{2} \quad \text{(10)}$$

in which w_T = lateral deflection.

Let $w_T = w_i + w$ .... (11)

in which w_i ° lateral deflection due to Q alone *(P = 0),* and w ° incremental lateral deflection. Then

$$EIw_i = -Q\frac{x^3}{12} + QL^2\frac{x}{16} \quad \text{(12)}$$

and $EIw_i'' = -\dfrac{Qx}{2}$ .... (13)

Subtract Eq. 13 from Eq. 10 and get

$$EIw'' + P(w + w_i) = 0 \quad \text{(14)}$$

which is the differential equation for a column with an initial lateral deflection w_i. Thus we have established that, even in this nonlinear problem, *the order of application of the loads P and Q is immaterial.* Therefore, one can apply Q first and solve for w_i, then add P and get w_T.

Now, to obtain the approximation in Eq. 9, it is only necessary to approximate w_i with a sine function:

$$w_i = \delta \sin\frac{\pi x}{L} \quad \text{(15)}$$

in which the coefficient d could be obtained by Fourier analysis or by minimizing the total potential energy. Inserting Eq. 15 into Eq. 14 and applying the boundary conditions of simple support yields

$$w = \frac{P}{P_{cr} - P}\,\delta \sin\frac{\pi x}{L} \quad \text{(16)}$$

Therefore $$w + w_i = \frac{\delta}{1 - \frac{P}{P_{cr}}} \sin \frac{\pi x}{L} \qquad (17)$$

$$= \Delta \sin \frac{\pi x}{L} \qquad (18)$$

Thus, the total midspan deflection =

$$\Delta = \frac{\delta}{1 - \frac{P}{P_{cr}}} \qquad (19)$$

from which EQ. 9 follows.

The advantage of Eq. 9 is that it has generally applicability. This is because w_i can always be represented by a Fourier series for any type of lateral loading, and the leading term will always behave as indicated by Eq. 9. Furthermore, the leading term will dominate as P approaches P_{cr}. This argument is due to Southwell (2). The method will next be applied to lateral and torsional stiffness of beams.

ANALYSIS OF BEAMS UNDER COMBINED TORSION AND BENDING

Consider a simply supported rectangular beam under constant bending moment in which ends are restrained against tipping, as shown in Fig. 3 (reaction torques shown). As to its lateral stiffness under a horizontal force, Q, at midspan and its torsional stiffness under an applied torque, T, at midspan (the ends are prevented from rotating about the x-axis), these stiffnesses should deteriorate to zero as the bending moment approaches the critical value for lateral-torisional buckling.

Again the order of application of the loads is immaterial since the differential equations and boundary conditions are linerar. Imagine the loads Q and T are applied first and produce the following intital deflections:

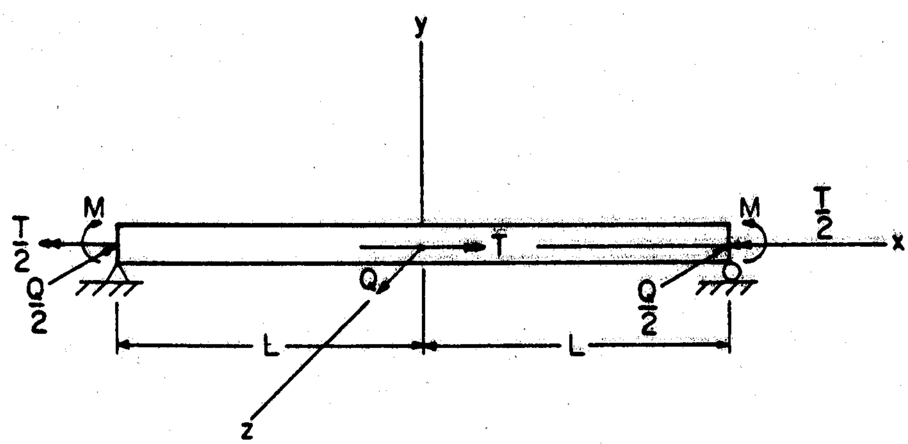

FIG. 3.–Simply Supported Beam

$$w_i = \delta \cos \frac{\pi x}{2L} \qquad (20)$$

$$\beta_i = \phi \cos \frac{\pi x}{2L} \qquad (21)$$

These deflections may be regarded as the leading terms of the exact Fourier series solution. From Ref. 3 the differential equations and boundary conditions are

$$EI_y w^{IV} + M\left(\beta + \phi \cos \frac{\pi x}{2L}\right)'' = 0 \qquad (22)$$

$$GK\beta'' - M\left(w + \delta \cos \frac{\pi x}{2L}\right)'' = 0 \qquad (23)$$

$$EI_y w''' + M\beta' = 0 \quad \text{at} \quad x = 0 \qquad (24)$$

$$w = 0 \quad \text{at} \quad x = L \qquad (25)$$

$$w' = 0 \quad \text{at} \quad x = 0 \qquad (26)$$

$$w'' = 0 \quad \text{at} \quad x = L \qquad (27)$$

$$GK\beta' - Mw' = 0 \quad \text{at} \quad x = 0 \qquad (28)$$

$$\beta = 0 \quad \text{at} \quad x = L \qquad (29)$$

and the solution is

$$w = \left(\frac{M^2}{M_{cr}^2 - M^2}\delta + \frac{M}{M_{cr}^2 - M^2}GK\phi\right)\cos\frac{\pi x}{L} \qquad (30)$$

$$\beta = \left(\frac{M^2}{M_{cr}^2 - M^2}\phi + \frac{M}{M_{cr}^2 - M^2}EI_y\frac{\pi^2}{4L^2}\delta\right)\cos\frac{\pi x}{L} \qquad (31)$$

Let D be the total lateral deflection, $w + w_i$, at midspan. From Eqs. 20 and 30,

$$\Delta = \frac{\delta}{1 - \left(\frac{M}{M_{cr}}\right)^2} + \frac{M}{M_{cr}^2} \cdot \frac{GK\phi}{1 - \left(\frac{M}{M_{cr}}\right)^2} \qquad (32)$$

Likewise let F be the total twist at midspan. From Eqs. 21 and 31

$$\Phi = \frac{\phi}{1 - \left(\frac{M}{M_{cr}}\right)^2} + \frac{M}{M_{cr}^2} \cdot \frac{\pi^2}{4L^2} \cdot \frac{EI_y\delta}{1 - \left(\frac{M}{M_{cr}}\right)^2} \qquad (33)$$

Now examine the deterioration of stiffness as M increases from zero to M_{cr}. At M = 0, the stiffnesses have initial values:

$$\delta = \frac{Q}{k_o} \qquad (34)$$

in which $k_o = \dfrac{48EI_y}{(2L)^3}$ = initial lateral stiffness .... (35)

and $\phi = \dfrac{T}{R_o}$ .... (36)

in which $R_o = \dfrac{2GK}{L}$ = initial torsional stiffness .... (37)

Examine first the case in which *T* and F are held constant. Subsituting Eq. 34 into Eq. 32 reduces it to the form

$$\Delta = \frac{Q}{k} + \Delta_o \qquad (38)$$

in which $k = k_o\left[1 - \left(\dfrac{M}{M_{cr}}\right)^2\right]$ .... (39)

= the lateral stiffness reduced by the primary bending load *M* and D_0 = the value of D at $Q = 0$. Note that lateral stiffness decays to zero at lateral buckling, and that the decay is *parabolic* rather than linear as was the case for simple columns above. D_0 also changes with *M;* it increases without bound at buckling.

Similarly, if *Q* is held constant, Eq. 36 reduces Eq. 33 to the form

$$\Phi = \frac{T}{R} + \Phi_o \qquad (40)$$

in which $R = R_o\left[1 - \left(\dfrac{M}{M_{cr}}\right)^2\right]$ .... (41)

= the reduced torsional stiffness, and $F_0 = F$ at $T = 0$.

Finally, if the load *Q* is acting at a distance *a* above the shear center, there is simultaneous torque and lateral load which are related by

$$T = Qa \qquad (42)$$

In this case the lateral stiffness is defined at the point of application of *Q,* i.e.

$$\delta + a\phi = \frac{Q}{k_o} \qquad (43)$$

defines k_o and $\Delta + a\Phi = \dfrac{Q}{k}$ .... (44)

defines *k.*

Combining Eqs. 32, 33, 42, 43, and 44, it can be shown that in this case

$$k = k_o \cdot \frac{1 - \left(\dfrac{M}{M_{cr}}\right)^2}{1 + \eta\dfrac{M}{M_{cr}}} \qquad (45)$$

in which $$\eta = \frac{\left(\frac{6}{\pi} + \frac{\pi}{2}\right)\gamma}{1 + 3\gamma^2} \qquad (46)$$

and $$\gamma = \frac{a}{L}\sqrt{\frac{EI_y}{GK}} \qquad (47)$$

i.e., the deterioration is even more rapid in this case if a is positive (toward the compression flange), and vice-versa.

The approximation error due to assuming Eqs. 20 and 21 is less than 5% by comparison with the exact solution. The exact solution is

$$w = -\frac{GK}{M}\left[\sqrt{\frac{EI_y}{GK}} \cdot \frac{T}{2M} + \frac{QM_{cr}}{\pi LM^2}\right]\left[\sin\lambda x - \tan\lambda L\cos\lambda x + \frac{\pi LM}{2M_{cr}}(x - L)\right] \qquad (48)$$

$$\beta = -\left[\sqrt{\frac{EI_y}{GK}} \cdot \frac{T}{2M} + \frac{QM_{cr}}{\pi LM^2}\right][\sin\lambda x - \tan\lambda L\cos\lambda x] + \frac{Q}{2M}(x - L) \qquad (49)$$

in which $$\lambda = \frac{M}{\sqrt{EI_y GK}} \qquad (50)$$

This solution yields the results if the following approximations are introduced:

$$\frac{\lambda^3 L^3}{3(\tan\lambda L - \lambda L)} \approx 1 - \left(\frac{M}{M_{cr}}\right)^2 \qquad (51)$$

$$\frac{\lambda L}{\tan\lambda L} \approx 1 - \left(\frac{M}{M_{cr}}\right)^2 \qquad (52)$$

both of which are accurate to within 5%.

If the primary beam load is not a constant bending moment, the results might be somewhat different. However, one can argue in the manner of Southwell (2) that the initial deformations can always be expanded in an eigenfunction series and the leading term will behave in the manner exhibited by Eqs. 32 and 33.

If one wishes to be more conservative, a good design recommendation would be the linear equations

$$k = k_o\left(1 - \frac{p}{p_{cr}}\right) \qquad (53)$$

and $$R = R_o\left(1 - \frac{p}{p_{cr}}\right) \qquad (54)$$

which are both conservative and simple to use.

SUMMARY

Readers are reminded that the onset of lateral-torsional buckling effectively reduces the lateral flexural rigidity and torsional rigidity of a beam to zero at buckling. They are cautioned against relying upon those rigidities to supply stiffness to ancillary portions of a structure when bending loads are present. The reduction factor was found to be parabolic in load, though a simple linear factor could conservatively be used in design.

APPENDIX I.–REFERENCES

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2. Southwell, R. V., "An Introduction to the Theory of Elasticity for Engineers and Physicists," Oxford University Press, Oxford, England, Ch. XIII, 1936.
3. Timoshenko, S. P., and Gere, J. M., "Theory of Elastic Stability," 2nd ed., McGraw Hill, New York, N.Y., 1961.

APPENDIX II.–NATATION

The following symbols are used in this paper:

a = distance from shear center to lateral force Q;
EI = flexural rigidity;
GK = torsional rigidity;
k = lateral stiffness. See Eqs. 5, 38, or 44;
k_o = initial lateral stiffness. See Eq. 2, 34, or 43;
L = length of column; half-length of bream;
M = bending moment about z axis:
P = axial force;
p = beam load;
Q = lateral force;
R = torsional stiffness. See Eq. 40;
R_o = initial torsional stiffness. See Eq. 36;
T = torque;
w = lateral deflection;
x, y, z = axes. See Fig. 3;
α = see Eq. 4;
β = angle of twist;
γ = see Eq. 47;
Δ = total midspan lateral deflection. See Eq. 3, 18, or 32;
δ = initial midspan lateral deflection. See Eq. 2, 15, or 20;
η = see 89. 46;
λ = see Eq. 50;
Φ = center twist. See Eq. 33; and
ϕ = initial center twist. See Eq. 21.