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EFFECT OF SIZE ON BENDING STRENGTH OF WOOD MEMBERS

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EFFECT OF SIZE ON BENDING STRENGTH OF WOOD MEMBERS

By
BILLY BOHANNAN,[2] Engineer

FOREST PRODUCTS LABORATORY[3]
FOREST SERVICE
U. S. DEPARTMENT OF AGRICULTURE

INTRODUCTION

For a long time it has been recognized that as a wood bending member increased in size its bending strength apparently decreased. Newlin and Trayer (4)[4] investigated this phenomenon in 1924 and related this decrease in strength to the depth of a bending member. This relationship, shown in figure 1, was developed from data obtained from beams having depths up to 12 inches.

With the growth of glued-laminated construction it became apparent that beams could be built to large sizes and that the strength-depth relationship was an important factor in design. In 1947, Dawley and Youngquist re-evaluated the strength-depth theory for wood beams and developed a new relationship. This relationship, also shown in figure 1, was developed from data obtained from beams having depths up to 16 inches. It was published by Freas and Selbo (3) in 1954 and is the present size-strength relationship recommended for use in structural design of wood bending members. Both equations shown in figure 1 are the strength ratios of a beam of depth, d, as compared to the strength of a beam having a depth of 2 inches.

As the size of glued-laminated beams continued to increase to depths exceeding 80 inches, it was questionable whether the size-strength relationship developed from data for beams with depths up to 16 inches could be extended to apply to such beams. There was a need to better define the size-strength phenomenon and the study discussed in this Research Paper was originated to investigate why a wood beam should decrease in strength as its size increased. The object of this study was to develop a mathematical theory to explain the size-strength relationship for wood beams and to check the theory using data from beams of several sizes.

This study is based on a statistical strength theory suggested by Weibull (6). Briefly, the basis for this theory is that there is a greater probability that a region of low strength will occur in a member of large volume than in a member of small volume. This region of low strength is assumed to cause complete failure of the member. This approach considers a volume-strength relationship rather than a depth-strength relationship, as has been used in the past. For a material that has a cumulative frequency distribution of strength (fig, 2) with certain characteristics, a relationship between strength and volume can be developed using statistical mathematics. The relationship for average modulus of

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[4] Underlined numbers in parentheses refer to Literature Cited at the end of this Paper.

rupture, $\bar{R}$, has the form of

$$\bar{R} = \frac{K}{f(V,P)}$$

where $\underline{K}$ is a constant or combination of different constants determined from the characteristics of the cumulative frequency distribution and $\underline{f(V,P)}$ is a function of the volume, $\underline{V}$, and method of loading, $\underline{P}$, Of the member.

This paper discusses the assumptions used in the statistical theory of strength, shows the application of the theory to wood bending members, and gives a comparison between theory and actual data for wood beams.

NOTATION

a--Distance between load points.
A--Aspect area (depth times span length).
b--Beam width.
B--Defined by equation (5).
d--Beam depth.
D--Standard deviation.
e--Naperian base.
F--Ratio of strength of beams having a depth, $\underline{d}$, to beams having a depth of 2 inches.
k--Material constant as defined by equation (2).
L--Span length.
m--Material constant as defined by equation (2).
n--Number of standard deviations.
n_o--A constant defined by equation (15).
P--Load on beam.
R--Modulus of rupture.
$\bar{R}$--Estimated average modulus of rupture of beams as defined by equation (7).
S--Theoretical cumulative frequency.
v--Coefficient of variation.
V--Volume of beam (portion within span).
Var(R)--Estimated variance of modulus of rupture. Refer to equation (9).
W--Defined by equation (16).
W_o--Material constant as defined in paragraph preceding equation (5).
Γ(--)--Denotes gamma function of parenthetical expression as defined in paragraph preceding equation (8).
σ--The tensile stress in a bending member.

STATISTICAL THEORY OF STRENGTH

The statistical theory of strength of materials used in this study is based on the "weakest link theory," as given by Weibull (6). This theory assumes that failure of a specimen will occur when the stress in the specimen is the same as the stress that would cause failure of the weakest element of volume if tested independently. This is probably a valid assumption for certain stress conditions and for a material that exhibits a brittle-type failure, For a ductile material, it seems likely that after failure of a small volume element, stresses would redistribute and complete failure would not occur until additional loads were applied, but for a brittle material, failure could propagate without additional loads being applied--hereinafter termed a cascade-type failure.

The "weakest link theory" of strength implies that the strength of a large member loaded in tension would be equal to the strength of the weakest of small pieces cut from the large member. To make this assumption valid, the small pieces must contain all material in the large member and the flaws or strength-reducing characteristics of the material may not be altered by the cutting of the small pieces. This same argument does not apply directly to members loaded in bending.

The stresses in a bending member are not uniform throughout the cross section: thus, the theory must be modified to account for the nonuniform stresses. In applying the theory to bending members, both the magnitude and location of the stresses must be considered. Nevertheless, regions of low strength are assumed to influence the strength of a bending member.

In the statistical strength theory, the cumulative frequency distribution of strength (fig. 2) is assumed to be a special type--one which lends itself to mathematical development of equations for predicting average strengths and standard deviations of strengths. In equation form, the theoretical cumulative-frequency curve is expressed as

$$S = 1 - e^{-B}$$

where S is the theoretical cumulative frequency; $B = \int f(\sigma)dV$ or the summation of a function of stress, $f(\sigma)$, times the volume element, dV, on which it acts; and e is the Naperian base (6). The function of stress, $f(\sigma)$, is a function expressing the local probability of failure. This is dependent on the local stress level and, hence, varies throughout the volume of a bending member. The functional form of $f(\sigma)$ is chosen so as to fit the distribution of strength and also to facilitate integration.

APPLICATION OF STATISTICAL STRENGTH THEORY OF WOOD BENDING MEMBERS

Development of Theory

Since final failure of a wood beam loaded in flexure is a tension failure and tension failure of wood exhibits a brittle rather than a ductile failure characteristic, it was theorized that the statistical strength theory would apply to wood bending members. By investigating the variability of tensile strength as developed in a bending member, it was believed that the size-strength effect of modulus of rupture of wood beams could be explained. This assumes that a cascade-type tension failure will occur when any element of volume fails in the tension region of a bending member. From this it follows that the tensile strength is dependent upon the volume of the member being evaluated.

In applying this theory to wood, it is assumed that the flaws or regions of low strength are not visible. This does not necessarily mean that the regions of low strength are microscopic in size. This use of the theory is not intended to explain strength reduction due to the commonly thought of strength-reducing characteristics such as knots and cross grain. It is intended to investigate the density or occurrence of regions of low strength in wood that otherwise appears to be free from strength-reducing characteristics.

One assumption of the "weakest link theory," as given by Weibull (6), is that the cumulative frequency distribution of strength (fig. 2) can be described by equation (1):

$$S = 1 - e^{-B} \tag{1}$$

where $B = \int f(\sigma)dV$. The form of the function, $f(\sigma)$, must be chosen so as to fit the cumulative distribution of strength and still describe the strength property being investigated. One form of the function that has these properties is $f(\sigma) = k\sigma^m$ where k and m are constants which influence the fit of equation (1) to the cumulative distribution of strength and ais the tensile stress in a bending member.

Before the mathematical expression for B can be evaluated, the stress distribution at failure must be known. For lack of a better theory, the linear stress theory as is assumed in calculating the modulus of rupture was used. Its simplicity for calculations made it an attractive basis from which to start, An attempt was made to apply. a plastic, nonlinear theory of stress at failure for wood beams as suggested by Bechtel and Norris (1), but this theory did not seem to have the desired properties and also was mathematically intractable.

The linear theory of stress is correct only when stresses are less than the proportional limit stress, but for this study it was believed sufficiently accurate in explaining stress distribution to failure. The hypothesis for this study is that the bending strength decreases as the size of wood members increases. From this hypothesis it follows that the strength of larger members may be such that it very nearly obeys the linear theory of stress: also, applying the “weakest link theory” to beams of small volume to predict the strength of larger beams implies that the lower strength values of small beams are being investigated. These lower strength members essentially obey the linear elementary theory all the way to failure. The error of using elementary theory is greatest for the strong pieces which probably do not greatly influence the prediction of strength for large members. It was believed, therefore, that little could be gained by using a nonlinear theory of stress which would introduce serious mathematical difficulties for very little improvement in the final results.

Using the linear theory of stress and a two-point method of loading (fig. 3), the development of an expression for B is as follows:

$$B = \int k\sigma^m \, dV \tag{2}$$

where k and m are material constants to be determined by fitting of equation (1) to cumulative frequency distributions of actual data, σ is the tensile stress in the member, or

$$\sigma = \frac{6Pxy}{bd^3}, \text{ for } 0 < x < \frac{L - a}{2}$$

and

$$\sigma = \frac{3P(L - a)y}{bd^3}, \text{ for } \frac{L - a}{2} < x < \frac{L + a}{2}$$

x and y are measured as shown in figure 3, and $dV = dzdydx$ where dz, dy, and dx are differential measurements of width, depth, and length, respectively. In these expressions

P--load on beam at failure,
a--distance between load points,
L--beam span length,
d--beam depth, and
b--beam width.

By substituting appropriate expressions into equation (2) and including appropriate limits of integration

$$B = 2\int_0^{\frac{L-a}{2}}\int_0^{\frac{d}{2}}\int_0^{b} k\left(\frac{6Pxy}{bd^3}\right)^m dzdydx + 2\int_{\frac{L-a}{2}}^{\frac{L}{2}}\int_0^{\frac{d}{2}}\int_0^{b} k\left(\frac{3P(L-a)y}{bd^3}\right)^m dzdydx \qquad (3)$$

Integration of equation (3) gives

$$B = \frac{(bdL)k}{2(m+1)^2}\left(1 + \frac{a}{L}m\right)\left(\frac{3}{2}\frac{P(L-a)}{bd^2}\right)^m \qquad (4)$$

An examination of the parts of this equation shows that

$V = bdL$ = volume of the member and

$R = \frac{3}{2}\frac{P(L-a)}{bd^2}$ = modulus of rupture calculated by linear theory

Substitution of these into equation (4) gives

$$B = \frac{kVR^m}{2(m+1)^2}\left(1 + \frac{a}{L}m\right)$$

Now if <u>k</u> and <u>m</u> are constants, the combination of terms $\frac{k}{2(m+1)^2}$ is a constant. By setting this combination equal to $\frac{1}{(W_o)^m}$, the expression for <u>B</u> for a beam under two-point loading with two equal concentrated loads symmetrically placed becomes

$$B = V\left(\frac{R}{W_o}\right)^m \left(1 + \frac{a}{L}m\right) \qquad (5)$$

This expression also applies to a center-loaded beam, for which a = 0. By substituting equation (5) into equation (1). the expression for the cumulative frequency distribution becomes

$$S = 1 - e^{-V\left(\frac{R}{W_o}\right)^m \left(1 + \frac{a}{L} m\right)} \tag{6}$$

The theoretical average modulus of rupture of beams, $\overline{R}$, that have a cumulative frequency distribution as given by equation (6) is calculated as follows:

$$\overline{R} = \int_0^\infty R dS = \int_0^\infty R e^{-V\left(\frac{R}{W_o}\right)^m \left(1 + \frac{a}{L} m\right)} \frac{Vm}{W_o}\left(1 + \frac{a}{L} m\right)\left(\frac{R}{W_o}\right)^{m-1} dR \tag{7}$$

To integrate, let

$$u = V\left(\frac{R}{W_o}\right)^m \left(1 + \frac{a}{L} m\right)$$

then,

$$du = \frac{Vm}{W_o}\left(1 + \frac{a}{L} m\right)\left(\frac{R}{W_o}\right)^{m-1} dR$$

Making these substitutions gives

$$\overline{R} = \frac{W_o}{\left[V\left(1 + \frac{a}{L} m\right)\right]^{1/m}} \int_0^\infty u^{1/m} e^{-u} du$$

The integral part of this expression is the gamma function of $\left(1 + \frac{1}{m}\right)$, usually designated as $\Gamma\left(1 + \frac{1}{m}\right)$. Values for the gamma function can be found in books of standard mathematical tables, Therefore, an expression for the relationship between average modulus of rupture and the volume and method of loading of a bending member is

$$\overline{R} = \frac{W_o \Gamma\left(1 + \frac{1}{m}\right)}{\left[V\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}} \tag{8}$$

The estimated variance of the modulus of rupture is found by

$$Var(R) = E(R^2) - \bar{R}^2$$

$\bar{R}$ is given by equation (8) and

$$E(R^2) = \int_0^m R^2 dS$$

By using the same substitution procedures as were used in developing equation (8), $E(R^2)$ is

$$E(R^2) = \left[\frac{W_o}{\left[V\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}}\right]^2 \Gamma\left(1 + \frac{2}{m}\right)$$

Therefore.

$$Var(R) = \left[\frac{W_o}{\left[V\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}}\right]^2 \Gamma\left(1 + \frac{2}{m}\right) - \left[\frac{W_o}{V\left(1 + \frac{a}{L} m\right)^{\frac{1}{m}}}\right]^2 \Gamma^2\left(1 + \frac{1}{m}\right) \qquad (9)$$

The standard deviation is the square root of the variance, therefore the standard deviation D is

$$D = \frac{W_o}{\left[V\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}} \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{\frac{1}{2}} \qquad (10)$$

If W_o and m are constants not dependent upon volume, then by statistical strength theory the average modulus of rupture of beams of any volume can be calculated from equation (8) and the confidence limits of modulus of rupture can be calculated using the standard deviation as given by equation (10). Examination of equations (8) and (10) show that both the average modulus of rupture and standard deviation of modulus of rupture are volume dependent.

Some additional properties of the statistical strength theory can be developed by letting the modulus of rupture R equal the average modulus of rupture $\bar{R}$ plus or minus some number n of standard deviations D, or

$$R = \bar{R} \pm nD \qquad (11)$$

Substituting equation (8) for $\bar{R}$ and equation (10) for D into (11) gives

$$\bar{R} \pm nD = \frac{W_o}{\left[V\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}} \left[\Gamma\left(1 + \frac{1}{m}\right) \pm n\left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{\frac{1}{2}}\right]$$

Rearrangement of this equation and taking each side of the equation to an m power gives

$$V\left(1 + \frac{a}{L} m\right)\left(\frac{\bar{R} + nD}{W_o}\right)^m = \left(\Gamma\left(1 + \frac{1}{m}\right) \pm n\left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{\frac{1}{2}}\right)^m \quad (12)$$

The left term in this equation is equal to the exponent of e in the cumulative distribution equation (6); thus, when the theoretical cumulative frequency S is zero, each side of equation (12) must equal zero. This will occur if

$$n_o = \frac{\Gamma\left(1 + \frac{1}{m}\right)}{\left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{\frac{1}{2}}} \quad (13)$$

and if

$$\bar{R} = n_o D \quad (14)$$

The standard deviation D in equation (14) can be written as $D = v\bar{R}$, where v is the coefficient of variation. Making this substitution into (14) gives

$$v = \frac{1}{n_o} \quad (15)$$

If m is a constant not dependent upon volume, then, by equation (13), n_o must also be a constant. Likewise, by equation (15), the coefficient of variation v must be a constant for beams of any volume.

Experimental Data Used to Evaluate Theory.

Data for four groups of clear, straight-grained Douglas-fir beams were used to evaluate the statistical theory. Details of the data are given in table 1. The moisture content of all specimens was

Table 1.--Data for Douglas-fir beams

Data group	Size			Number of specimens	Average modulus of rupture	Standard deviation	Coefficient of variation	How loaded	Origin of data
	Width	Height	Span length						
	In.	In.	In.		P.s.i.	P.s.i.	Pct.		
A	1	1	14	343	13,250	778	5.9	Center point	FPL standard strength data.
B	1	1	18	210	13,350	975	7.3	.........do.........	Controls from duration of load study. Forest Products Lab. Rpt. 1916.
C	2	2	28	1,418	12,330	1,056	8.5	.........do.........	FPL standard strength data.
D	5.2	12	162	85	9,520	679	7.15	Two point, 18 inches between loads	Control beams. Forest Products Lab. Rpt. 1687 (laminated beams)
Combined A+B	1	1	[1]16	553	13,290	863	6.5	Center point	

[1]Assumed average size of groups A and B.

approximately 12 percent. To eliminate effect of specific gravity on strength values, all values were corrected to a constant specific gravity of 0.48 by means of first-order regression equations calculated for each set of data. This correction for specific gravity was necessary because for small volume changes the effect of specific gravity on strength was greater than the effect of volume change.

After strength values were adjusted for effects due to specific gravity, the average properties for groups A and B were approximately equal. The beam size of group A was 1 by 1 by 14 inches and that of group B was 1 by 1 by 18 inches. It was believed that the statistical strength theory was not sensitive enough to indicate differences in strength for such small volume changes. Therefore, the two groups were combined into one group. The assumed beam size for the combined group was 1 by 1 by 16 inches.

Determination of m and W_o

To complete the relationship between strength and volume of bending members, estimates of m and W_o are required. With constant values for m and W_o substituted into the cumulative frequency distribution,

$$S = 1 - e^{-V\left(1 + \frac{a}{L} m\right)\left(\frac{R}{W_o}\right)^m} \tag{6}$$

the exponent of e is then dependent only on volume and method of loading. Volume independence of m and W_o is assured if equation (6) will then approximately describe the observed cumulative frequency distribution of modulus of rupture for groups of beams of different volumes.

By letting

$$\frac{1}{W^m} = \frac{V\left(1 + \frac{a}{L} m\right)}{\left(W_o\right)^m} \tag{16}$$

the cumulative frequency distribution becomes

$$S = 1 - e^{-\left(\frac{R}{W}\right)^m} \tag{17}$$

Now, by equation (16), the parameter $\underline{W}$ is volume dependent. Also, when the exponent of $\underline{e}$ is unity in equation (17) the probability of failure is 0.632. For the exponent to be unity, $\underline{W}$ must equal $\underline{R}$. Thus, $\underline{W}$ is equal to the modulus of rupture of beams of a given volume that have a probability of failure of 0.632. Values of $\underline{W}$ can be estimated from the observed cumulative frequency curves for different sets of data, figure 4. By substituting appropriate values for $\underline{W}$, $\underline{V}$, and $\frac{a}{L}$ for the three sets of data, given in table 1 and figure 4, into equation (16), three equations having two unknowns, $\underline{m}$ and $\underline{W_o}$, are generated. If $\underline{m}$ and $\underline{W_o}$ are constants, then these three equations have one common solution.

The best common estimates of $\underline{m}$ and $\underline{W_o}$ can be determined graphically from a plot of equation (16) for each set of data. Such a graph is shown in figure 5. These curves do not all intersect at a common point, but they do indicate that an approximate fit of the data could be expected with an $\underline{m}$ of 24 and $\underline{W_o}$ of 15,350.

Comparison of Theory to Observed Data

Using the values of m = 24 and W_o = 15,350, the theoretical cumulative frequency distribution of modulus of rupture becomes

$$S = 1 - e^{-V\left(1 + \frac{24a}{L}\right)\left(\frac{R}{15,350}\right)^{24}}$$

A graph of this equation and of the observed cumulative frequency distributions for each set of data are shown in figures 6, 7, and 8. It is evident from these figures that there is some disagreement between theory and observed data. Complete agreement was not expected since values of $\underline{m}$ and $\underline{W_o}$ were chosen so as to best fit the three sets of data As shown by figure 5, common values did not uniquely fit the three sets of data.

Another reason to expect some disagreement in the fit of observed and theoretical cumulative distributions of modulus of rupture is because the theory indicates that the coefficient of variation of modulus of rupture is a constant. This is shown by equations (12) through (15). As given in table 1, the coefficient of variation is not a constant for the three sets of data, but it does not seem to be particularly dependent on volume; thus it is believed that the coefficient of variation could be assumed to be a constant without too much error, With m = 24, the theoretical coefficient of variation is 5.2 percent. This is lower than any of the observed coefficient of variation values for the three sets of data.

These differences between theory and data indicate that possibly some of the assumptions used in developing the theory are not justified or that some modifications of the theory need to be made before applying the theory to wood bending members.

APPLICATION OF MODIFIED STATISTICAL STRENGTH THEORY TO WOOD BEAMS

The statistical strength theory that relates strength to volume of a wood bending member as discussed in previous sections of this paper, only approximately agrees with observed data; thus, it seems possible that better agreement might be obtained by some modification of the theory.

The "weakest link theory" of strength is based on the assumptions of "links" being in series. Another way of considering "links" is that they act in parallel, Weibull (6) concludes that the results will be the same regardless of how the links are coupled. His conclusion is based on the assumption that cascade failure of a member will occur with failure of any link or element of volume. For a wood bending member, this assumption may not be valid. Whether the links are in series or parallel may influence how the theory applies to wood. bending members.

Considering the actual wood beam, elements of length are in the direction of bending stresses, thus they are in series. A failure cascading at any cross section throughout the length would result in complete failure of a beam. This makes length a factor in the statistical theory. The elements of width and depth are perpendicular to the direction of stresses, thus these elements are connected in parallel, Even though they are both in parallel, they react differently in supporting a moment on a beam. A cascading failure may not always occur across the width but it is believed that a cascading failure can occur across the depth. A better understanding of this is obtained by considering the sequence of failure of a wood beam and the stresses imposed on elements of width and depth.

As a beam is loaded, failure may start without cascade and additional loading is then required before complete failure. Nevertheless, it has been observed that failure occurs completely across the width of a beam before the failure propagates much in the direction of depth. This does not necessarily mean that failure across the width will always start at the outer face of the beam. It can occur at a region of low strength away from the outer face. During evaluation of a beam having nineteen 9/16-inch-thick laminations, failure was observed to start in the third lamination.

A cascade failure is not usually associated with beams tested in a laboratory. Most laboratory testing machines are actually deflection machines on which load is recorded as a function of the deflection of the beam. Once failure starts, the load is reduced because the section modulus of the beam is reduced and less load is required for the same beam deflection. Under this type of loading complete failure is not sudden, but it is believed that, for a beam under dead load, once failure has occurred across the width there would be no stopping of a cascade across the depth.

Based on this observed sequence of failure of wood beams, the assumption of a cascade across the width is rather implausible especially if the region of initial failure is small, but after the beam has failed across the width, it is plausible that cascade will occur across the depth. This is further clarified by considering what happens as the beam starts to fail. When an element of width fails, the stress in the remaining elements is increased by $\frac{N}{N-1}$, where N is the number of elements. This increase may or may not cause failure of the next weakest element, thus cascade may not occur, Now considering elements of depth, these elements are also in parallel but react much differently than do elements of width. Once a section has failed across the width, a region of depth also has failed or a link in the parallel links of depth has failed. The increase in stress on the remaining links is greater than $\frac{N}{N-1}$ as it was for links of width. When a link of depth fails, the cross section is changed and thus the section modulus is reduced. The result is an unchanged moment to be supported by a decreased section modulus. This could cause a stress increase sufficient to fail adjacent links which will lead to a cascade.

Considering again the elements of width, the statistical theory implies that as the width increases the probability of the occurrence of a region of low strength increases, but as the width increases the probability of cascade across the width decreases. The net effect of these counteracting probabilities may be negligible. Therefore, it is believed that width should not be considered in the application of the "weakest link theory" to wood beams.

Another rationalization for not considering width in the theory can be drawn from equation (8), the formula for predicting average modulus of rupture

$$\overline{R} = \frac{W_o \Gamma\left(1 + \frac{1}{m}\right)}{\left[V\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}}$$

This shows that the average modulus of rupture is equal to a constant divided by a function of volume. With the function of volume equal to the actual volume of a beam, this equation indicates that the strength of a beam loaded on edge, or smallest dimension of cross section, is the same as the strength of a beam of equal size loaded on its face, or largest dimension of cross section. This is not consistent with the connotative idea of size effect on modulus of rupture of a wood beam thus it seems possible that size effect on modulus of rupture is dependent upon the length and depth of a beam and independent of the width,

Tucker (5), in his discussion of the "weakest link theory" and his application of a statistical strength theory to concrete beams, concluded that the size effect on modulus of rupture was independent of the beam width.

By assuming that width is not a factor in the statistical strength theory, derived expressions for cumulative frequency distribution, modulus of rupture, and standard deviations of modulus of rupture are:

$$S = 1 - e^{-A\left(\frac{R}{W_o}\right)^m \left(1 + \frac{a}{L} m\right)} \tag{18}$$

$$\overline{R} = \frac{W_o \Gamma\left(1 + \frac{1}{m}\right)}{\left[A\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}} \tag{19}$$

and

$$D = \frac{W_o}{\left[A\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}} \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{\frac{1}{2}} \tag{20}$$

The only change in these expressions from those as given by equations (6), (8), and (10) is that the volume term **V** in the latter expressions was replaced by the aspect area (depth times length) A. The assumed constants m and W_o have the same meaning as they did in equations (6), (8), and (10) but they will have different values.

The appropriate values for m and W_o can be determined by the same procedure as was previously discussed. With the width eliminated, equation (16) becomes

$$\frac{1}{W^m} = \frac{A\left(1 + \frac{a}{L} m\right)}{\left(W_o\right)^m} \quad (21)$$

Values of A, W, and $\frac{a}{L}$ for different sets of data are given in table 1 and figure 4. Substituting these values into equation (21) again generates three equations having two unknowns, m and W_o. As before, the equations should have one common solution if m and W_o are constants.

Graphs of equation (21) for each set of data are shown in figure 9. This figure shows that the three curves do very nearly intersect at a common point where m = 18 and W_o = 15,900.

A comparison of figures 5 and 9 shows that there is a better agreement for constant values of m and W_o by not considering width in the statistical theory than when width is considered.

Using m = 18 and W_o = 15,900 in equation (18), the cumulative frequency distribution is

$$S = 1 - e^{-A\left(\frac{R}{15,900}\right)^{18}\left(1 + \frac{18a}{L}\right)}$$

Graph of this equation and observed cumulative frequency distribution of modulus of rupture for the three sets of data given in table 1 are shown in figures 10, 11, and 12. As shown in these figures, there is reasonably close agreement between theory and data. Complete agreement was not expected due to the theoretical requirement of a constant coefficient of variation, as has been previously discussed. With m = 18, the theoretical coefficient of variation is 6.83 percent. This is approximately equal to the coefficient of variation for beam groups A plus B and D that have values of 6.5 and 7.15 percent, respectively. The greatest disagreement between theory and data is for group C beams (fig. 11) that has a coefficient of variation of 8.3 percent.

A comparison between figures 6, 7, and 8 and figures 10, 11, and 12 shows that considerably better agreement between theory and observed data is obtained by not considering width of beams in the theory.

EXPERIMENTAL EVALUATION OF EFFECT OF SIZE ON MODULUS OF RUPTURE

To experimentally investigate the effect of size, particularly width, on modulus of rupture, 50 Douglas-fir beams of four different sizes were evaluated under loads placed at one-third the span from each reaction $(a = \frac{L}{3})$. The specimen sizes and average strength properties are given in table 2.

Table 2.--Data on four sizes of Douglas-fir beams

Data group	Width	Depth	Span length	Number of specimens	Average modulus of rupture[1]
	In.	In.	In.		P.s.i.
F	1	1	21	34	12,100
G	6	1	21	42	12,100
H	2.75	0.6	12.75	43	12,400
I	1	1	30	28	11,700

[1]Values rounded to nearest 100 p.s.i.

Specimens were sawed from nomnal 2- by 8-inch material and were either side or end matched. Group G was end matched to groups F, H, and I which were side matched. After testing, some specimens were discarded because of localized areas of cross grain. This resulted in having less than 50 specimens per group as indicated in table 2.

The sizes were chosen to investigate the effects of different dimensions on strength of the beams. Group F, having 1- by 1- by 21-inch dimensions (21-cubic inches volume) was chosen as a control size and the sizes of other groups were chosen to evaluate the effect of different changes from this size on strength. Group G, having 6 inch width, 1 inch depth, and 21 inch length, was chosen to investigate the effect of width; group H, having 2.75- by 0.6- by 12.75-inch dimensions (21 cu. in. volume), was chosen to investigate effect of volume; and group I, having 1- by 1- by 30-inch dimensions, was chosen to investigate effect of length.

The average values of modulus of rupture (table 2) are approximately the same for all groups, but the small differences in strength may be significant. Small differences were expected because of the small sizes of beams evaluated and the small changes in sizes.

The average modulus of rupture for groups F and G are equal to the nearest 100 pounds per square inch which indicates no effect of a change of width from 1 to 6 inches. The modulus of rupture of group **H** is larger than that for group F, both groups having equal volume. This indicates that the increase was possibly caused by the decrease in length and depth. The modulus of rupture for group I is less than that for group F, which indicates increase in length caused a decrease in strength. These conclusions are based on very limited data but they do follow the hypothesis of modulus of rupture being dependent on length and depth but independent of width.

ESTIMATED AVERAGE MODULUS OF RUPTURE

The estimated average modulus of rupture of wood beams is given by equation (19)

$$\overline{R} = \frac{W_o}{\left[A\left(1 + \frac{a}{L} m\right)\right]^{\frac{1}{m}}} \Gamma\left(1 + \frac{1}{m}\right) \qquad (19)$$

A graph of this equation is shown in figure 13. In this equation, A is the product of depth and length, W_o = 15,900, and m = 18. The term $\left(1 + \frac{a}{L} m\right)$ accounts for method of loading of the beam. This term applies to two-point loading with two equal concentrated loads symmetrically placed with a the distance between load points. For center-loaded beams where a is zero, this term becomes unity; thus the equation also applies to center-loaded beams.

This equation shows that the modulus of rupture is dependent upon both depth times length and method of loading, The difference between a beam loaded at two points and a beam of equal volume loaded at the center is

$$\left(1 + \frac{a}{L} m\right)^{\frac{1}{m}}$$

With m = 18, the modulus of rupture of a center-loaded beam is 11.4 percent greater than the modulus of rupture of a similar beam loaded at third-span points, Tucker (5) found this difference to be 11.7 percent in his analysis.

A comparison of beams of different volumes shows that if the depth and length are both decreased by one-half, then the modulus of rupture increases by 8 percent. This agrees reasonably well with data on Douglas-fir beams published by Comben (2). His data show the strength of beams 1 inch high to be approximately 8.4 percent greater than the strength of beams 2 inches high. His beams were loaded at third-span points and they had a constant span-depth ratio of 16 to 1. Our data show that the average modulus of rupture of the 1- by 1- by 16-inch beams is 7.7 percent greater than the average modulus of rupture of the 2- by 2- by 28-inch beams. Theoretically this ratio is 7.2 percent.

The lower limiting value of modulus of rupture as calculated by equation (19) is zero for beams of infinite volume. It seems possible that there should exist a lower limit of modulus of rupture, other than zero, for beams of large volume. In development of the theory an attempt was made to include such a lower limit. This inclusion of a lower limit of modulus of rupture in the statistical strength theory was mathematically possible but a unique value could not be determined from actual data, thus, the idea of a lower limit was rejected.

Further evaluation of equation (19) shows that for a beam whose depth and length are three times greater than any beam presently in service, say depth = 21 feet and length = 400 feet, the theoretical modulus of rupture is approximately 7,000 pounds per square inch. The equation also shows that for a beam of unit volume, the theoretical modulus of rupture is approximately 15,400 pounds per square inch. The practical range of sizes of wood beams is well within these limits.

COMPARISON OF THEORETICAL AND EXPERIMENTAL MODULUS OF RUPTURE

Figure 14 shows the comparison of the theoretical and observed average modulus of rupture for beams having different volumes and methods of loading. The solid curve is the calculated values from equation (19). The broken line represents the average modulus of rupture. equation (19), minus two times the standard deviation, equation (20). Points A+B, C, and D are for the groups of data used to develop the theory (table 1). Points F, H, and I are data given in table 2. Point E is for a group of 237 Douglas-fir beams that were loaded at third-span points. The size of these beams was 1-3/4 by 2 by 45 inches. The points not coded are modulus of rupture values of individual beams. The data for all beams were adjusted to 12 percent moisture content and to 0.48 specific gravity.

Points marked L on figure 14 are modulus of rupture values of three beams having a cross section of 9 inches by 31-1/2 inches that were tested under 2-point loading over a 48-foot span with 10 feet between load points. These data are results of some recent tests as yet unpublished. These beams were laminated from clear straight-grained Douglas-fir that had an average specific gravity of approximately 0.48 and an average moisture content of approximately 12 percent.

Reasonably good agreement is shown between theory and observed data.

COMPARISON OF OLDER SIZE-STRENGTH THEORIES AND STATISTICAL STRENGTH THEORY

The older size–strength theories of bending strength of wood beams are shown in figure 1. Both of these theories relate the apparent decrease in strength to the depth of the beam The equations shown in figure 1

$$F = 1.07 - 0.07\sqrt{\frac{d}{2}}$$

and

$$F = 0.625\left(\frac{d^2 + 143}{d^2 + 88}\right)$$

are the ratios of the modulus of rupture of a beam having a depth d to that of a beam having a depth of 2 inches.

The statistical strength theory as given in this paper relates the bending strength to depth and length of the beam and method of loading the beam. Or by equation (19)

$$\bar{R} = \frac{W_o \Gamma\left(1 + \frac{1}{m}\right)}{(dL)^{\frac{1}{m}}\left(1 + \frac{am}{L}\right)^{\frac{1}{m}}}$$

To reduce this equation so that a comparison of the strength of beams of different depths can be made, certain conditions must be assumed. By assuming $\frac{a}{L}$ is a constant, or same method of loading, and assuming a constant span–depth ratio for beams being compared, the ratio of the strength of a beam having depth d to that of a beam having a depth of 2 inches is

$$F = \frac{R_d}{R_2} = \left(\frac{2^2}{d^2}\right)^{\frac{1}{m}}$$

With m = 18

$$F = \left(\frac{2}{d}\right)^{\frac{1}{9}}$$

A graph comparing the three strength ratio equations is shown in figure 15.

SUMMARY AND CONCLUSIONS

A statistical theory of strength of materials based on the "weakest link" theory as suggested by Weibull (6) was used to develop a size-strength relationship for the bending strength of wood beams. For beams loaded with two concentrated loads symmetrically placed, the derived expression for the cumulative frequency distribution of modulus of rupture is

$$S = 1 - e^{-V\left(1 + \frac{am}{L}\right)\left(\frac{R}{W_o}\right)^m} \qquad (6)$$

for the estimated average. modulus of rupture is

$$\bar{R} = \frac{W_o \Gamma\left(1 + \frac{1}{m}\right)}{\left[V\left(1 + \frac{am}{L}\right)\right]^{\frac{1}{m}}} \qquad (8)$$

and for the standard deviation of modulus of rupture is

$$D = \frac{W_o}{\left[V\left(1 + \frac{am}{L}\right)\right]^{\frac{1}{m}}} \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{\frac{1}{2}} \qquad (10)$$

where V = volume of beam, a = distance between load points, L =beam length, Γ(----) =gamma function of terms enclosed by parentheses, R = modulus of rupture, and $\underline{m}$ and $\underline{W_o}$ are material constants whose values are determined by fitting of equation (6) to observed data.

Data for three groups of Douglas-fir beams were used to check the theoretical relationships. Estimated values of $\underline{m}$ and $\underline{W_o}$ were determined but they did not uniquely fit the three sets of data; thus, the theoretical relationships only approximately agreed with the observed data.

The statistical theory was reevaluated with the conclusion that width of the beams should not be included in the theoretical relationships. With width not included, the theoretical relationships are

$$S = 1 - e^{-A\left(1 + \frac{am}{L}\right)\left(\frac{R}{W_o}\right)^m} \qquad (18)$$

$$\bar{R} = \frac{W_o \Gamma\left(1 + \frac{1}{m}\right)}{\left[A\left(1 + \frac{am}{L}\right)\right]^{\frac{1}{m}}} \qquad (19)$$

and

$$D = \frac{W_o}{\left[A\left(1 + \frac{am}{L}\right)\right]^{\frac{1}{m}}} \left[\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right]^{\frac{1}{2}} \tag{20}$$

The only change from equations (6), (8), and (10) to equations (18), (19), and (20) is that the volume term V, in the former equations was changed to the aspect area, A, depth times length, in the latter. Values of m and W_o were determined to be 18 and 15,900 pounds per square inch, respectively, and reasonably close agreement between theory and data was obtained. The average modulus of rupture was quite accurately predicted for Douglas-fir beams having sizes ranging from 1 inch deep by 14 inches long to 31-1/2 inches deep by 48 feet long.

Summarizing, the results indicate:

1. An increase in depth or an increase in length will cause a decrease in average modulus of rupture and standard deviation of modulus of rupture of wood beams.

2. Beams loaded with two equal concentrated loads, symmetrically placed, will have a lower average modulus of rupture and standard deviation of modulus of rupture than beams of equal volume loaded at midspan.

3. The modulus of rupture and standard deviation of modulus of rupture are independent of the beam width.

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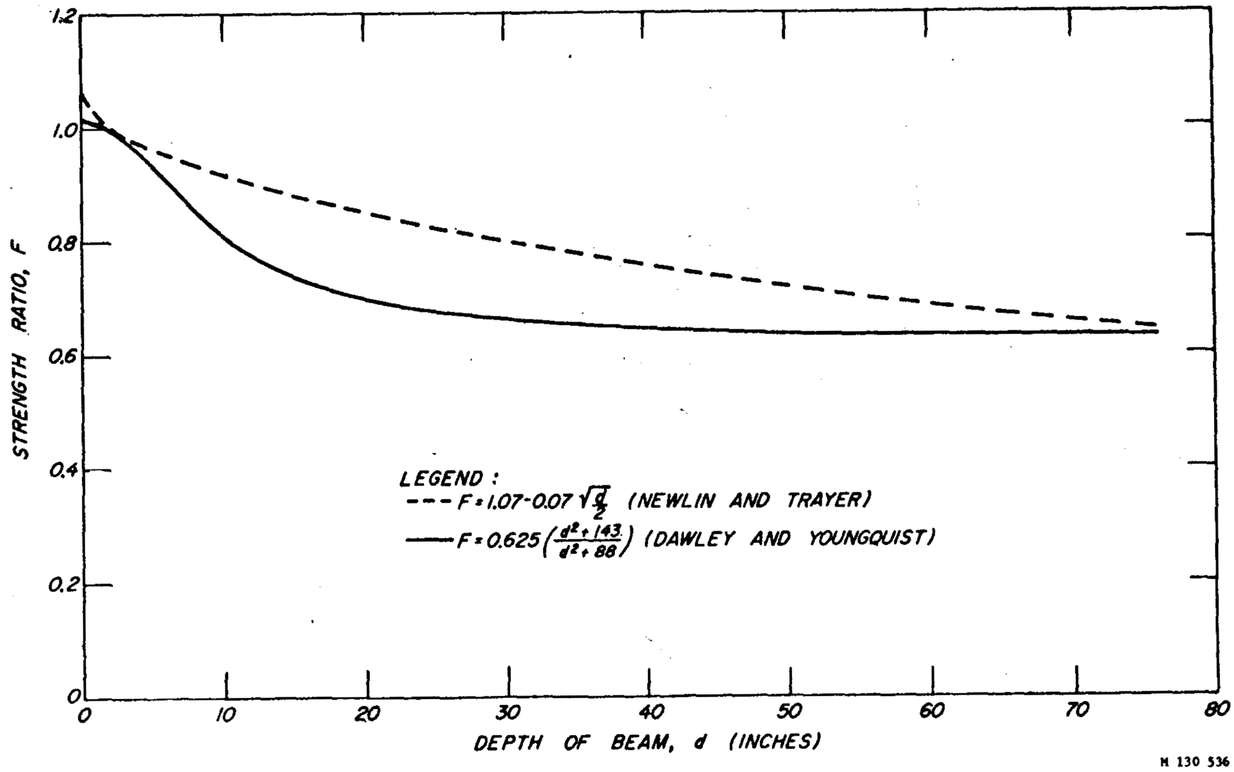

Figure 1.--Strength ratio, F, of wood beams having a depth, d, to beams having a depth of 2 inches.

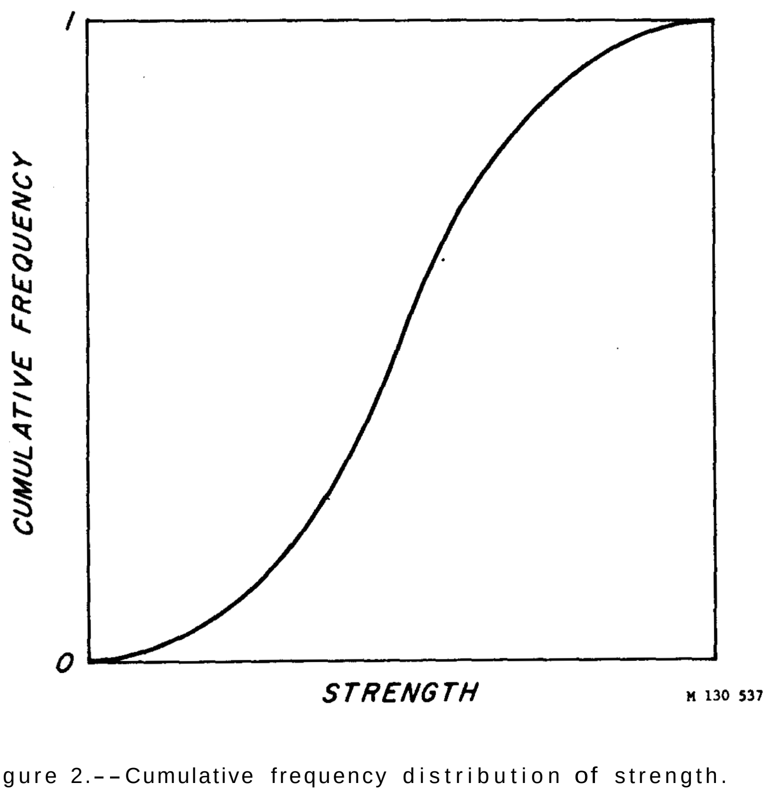

Figure 2.--Cumulative frequency distribution of strength.

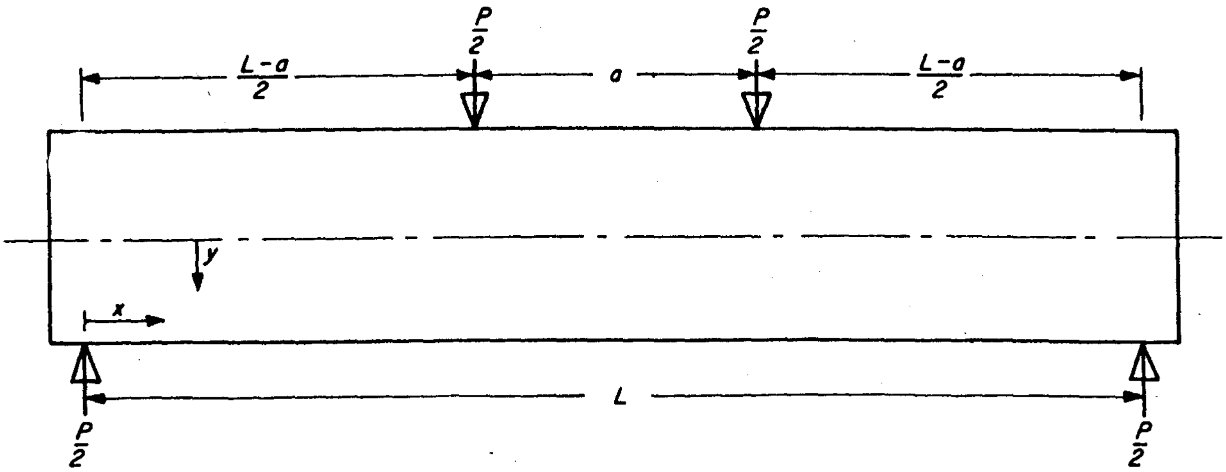

Figure 3.--Beam under two-point loading, showing rotation used in the derivation of theoretical formulas and expressions.

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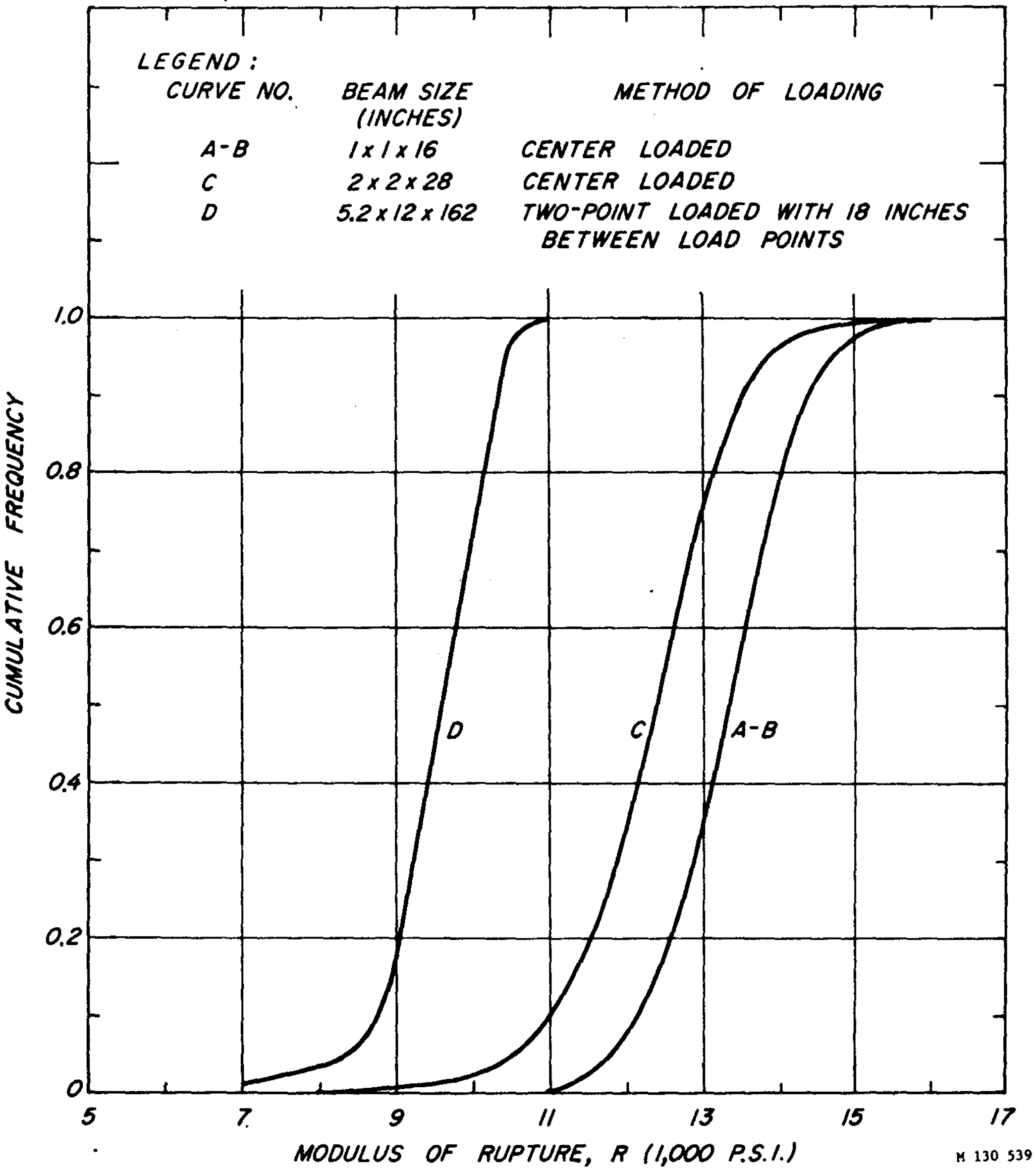

Figure 4.--Cumulative frequency distribution of modulus of rupture observed in three sets of data for Douglas-fir beams. Refer to table 1 for details.

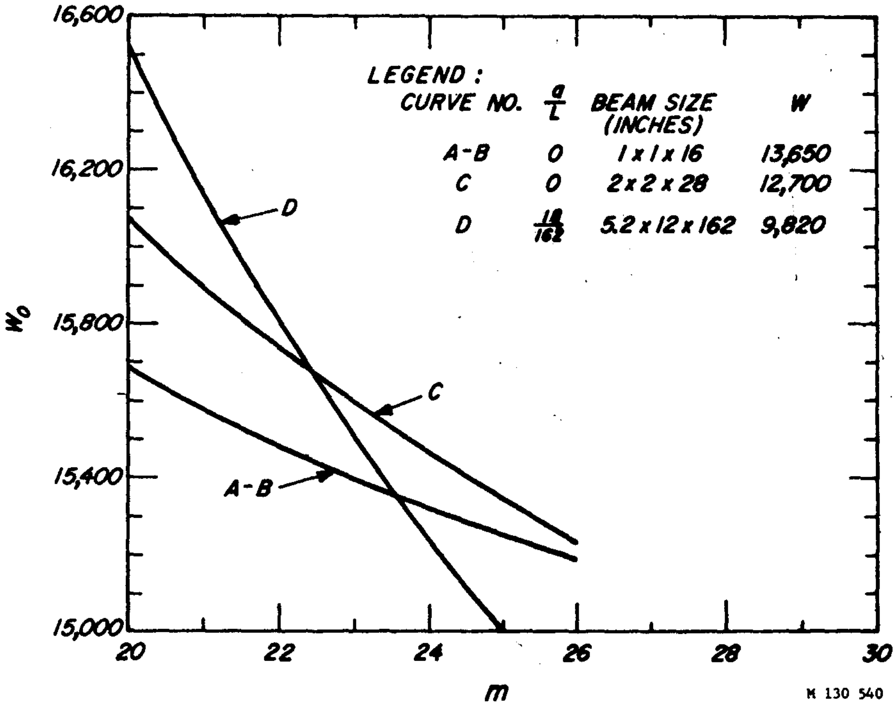

Figure 5.--Plot of equation $\left(\frac{W_o}{W}\right)^m = V\left(1 + \frac{a}{L}m\right)$, for three sets of data. V = volume of beam (portion within span); other values are given in legend.

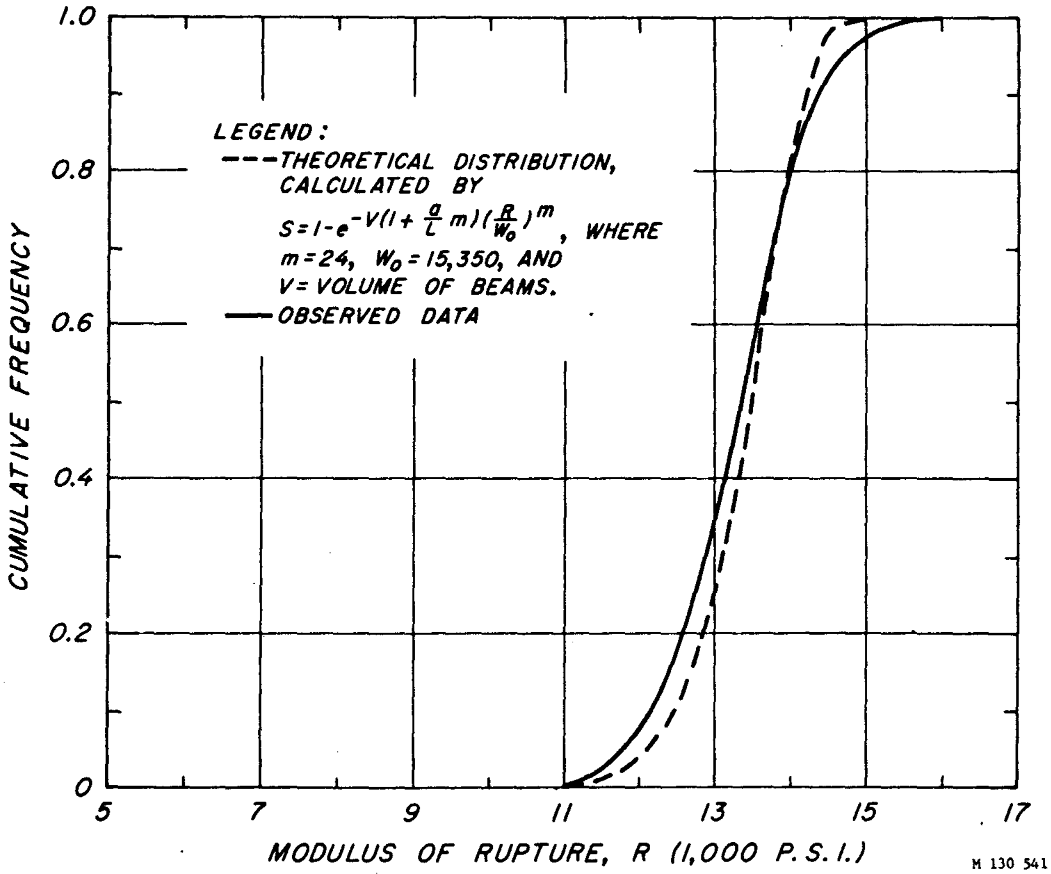

Figure 6.--Cumulative frequency distribution of modulus of rupture of beams 1 by 1 by 16 inches in size, center loaded. Theoretical calculations based on total volume of beams within span.

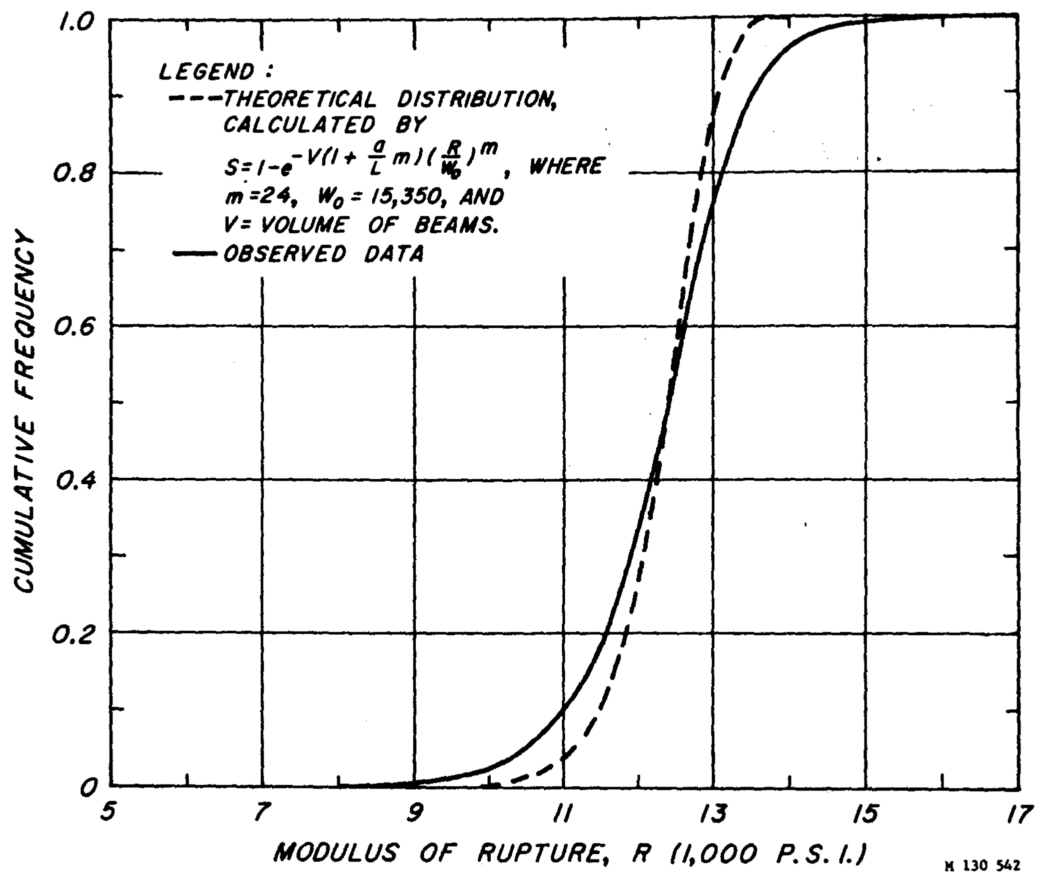

Figure 7.--Cumulative frequency distribution of modulus of rupture of beams 2 by 2 by 28 inches, center loaded. Theoretical calculations based on total volume of beams within span.

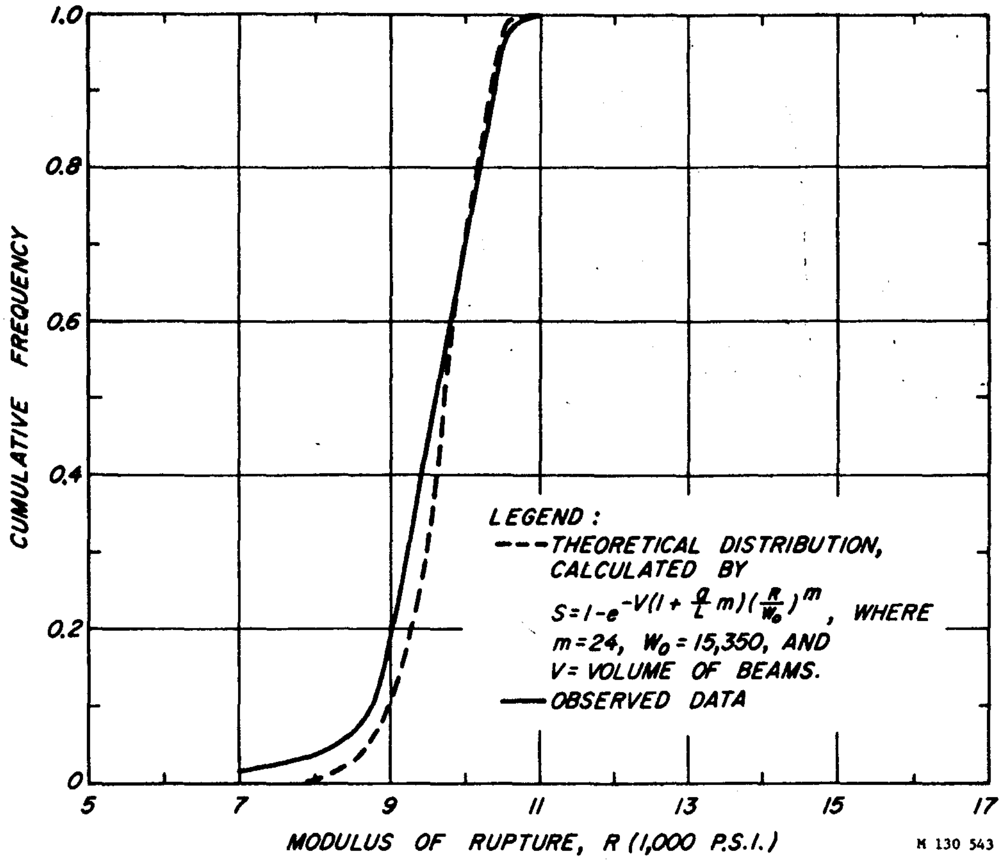

Figure 8.--Cumulative frequency distribution of modulus of rupture of beams 5.2 by 12 by 162 inches, two-point loaded, with 18 inches between load points. Theoretical calculations based on total volume of beams within span.

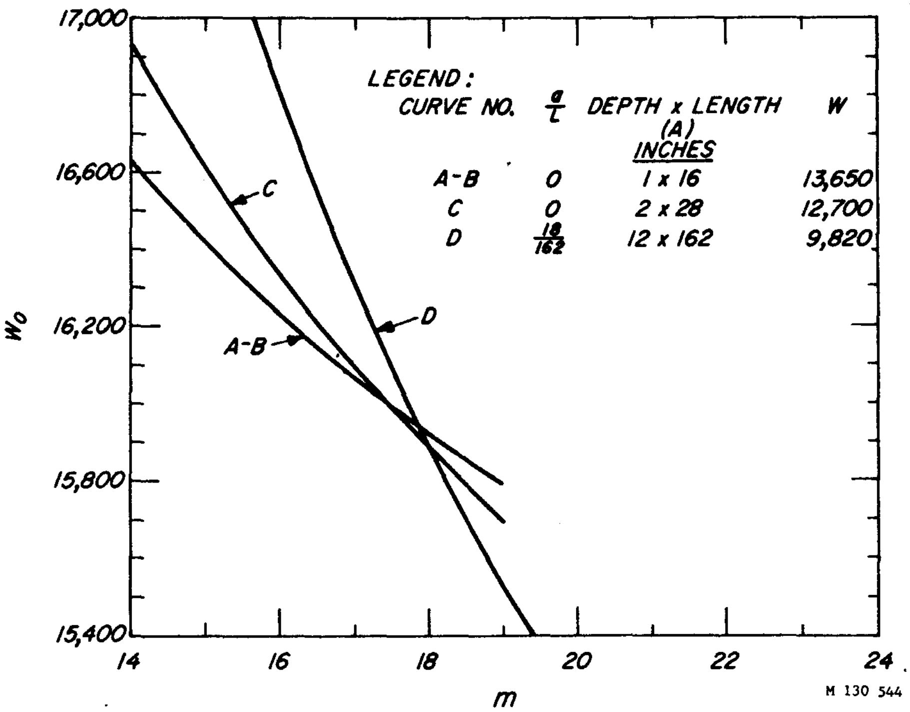

Figure 9.--Plot of equation $\left(\frac{W_o}{W}\right)^m = A\left(1 + \frac{a}{L} m\right)$ for three sets of data. Aspect area, A, equals depth times length. Other values are given in legend.

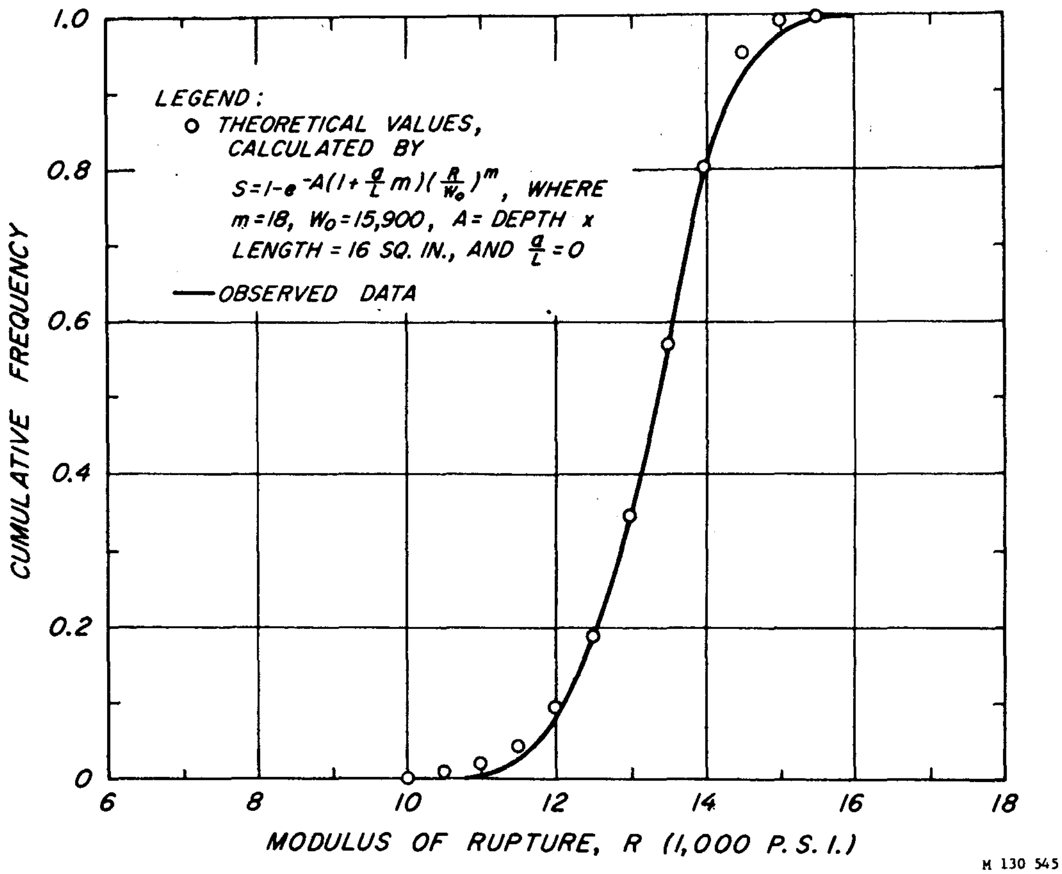

Figure IO.--Cumulative frequency distribution of modulus of rupture of beams 1 by 1 by 16 inches, center loaded. Theoretical calculations based on aspect area of beams within span.

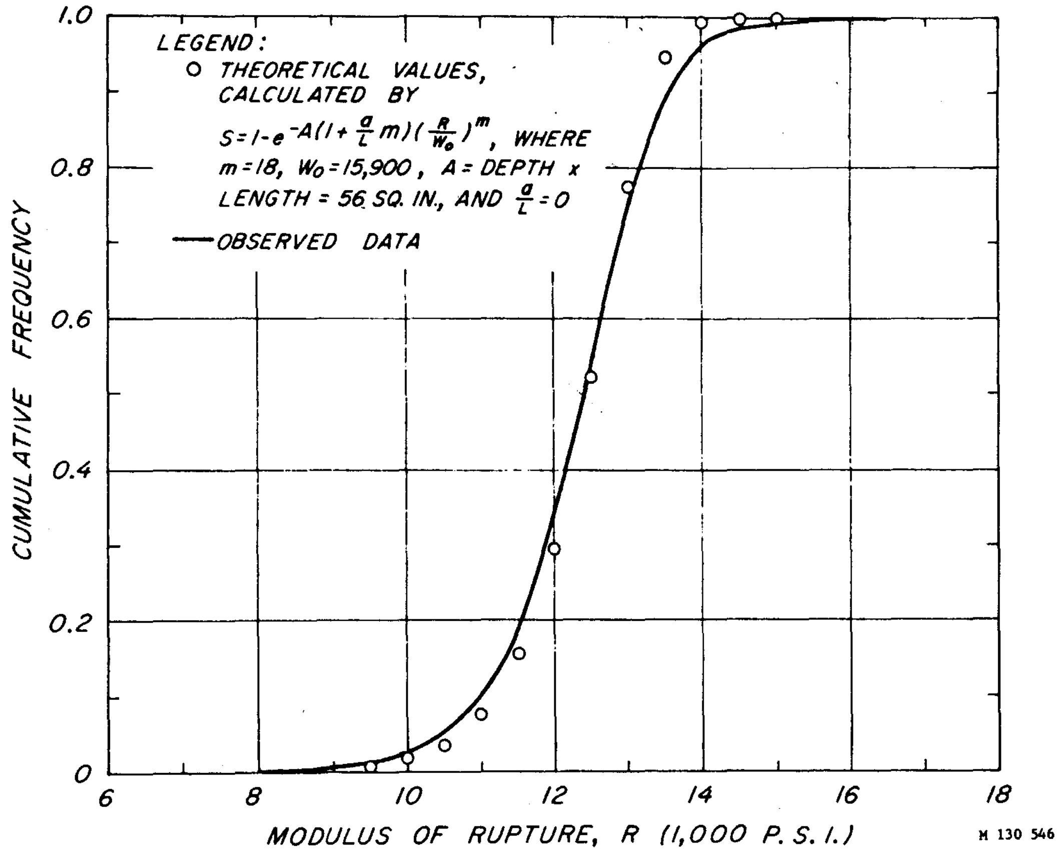

Figure 11.--Cumulative frequency distribution of modulus of rupture of beams 2 by 2 by 28 inches, center loaded. Theoretical calculations based on aspect area of beams within span.

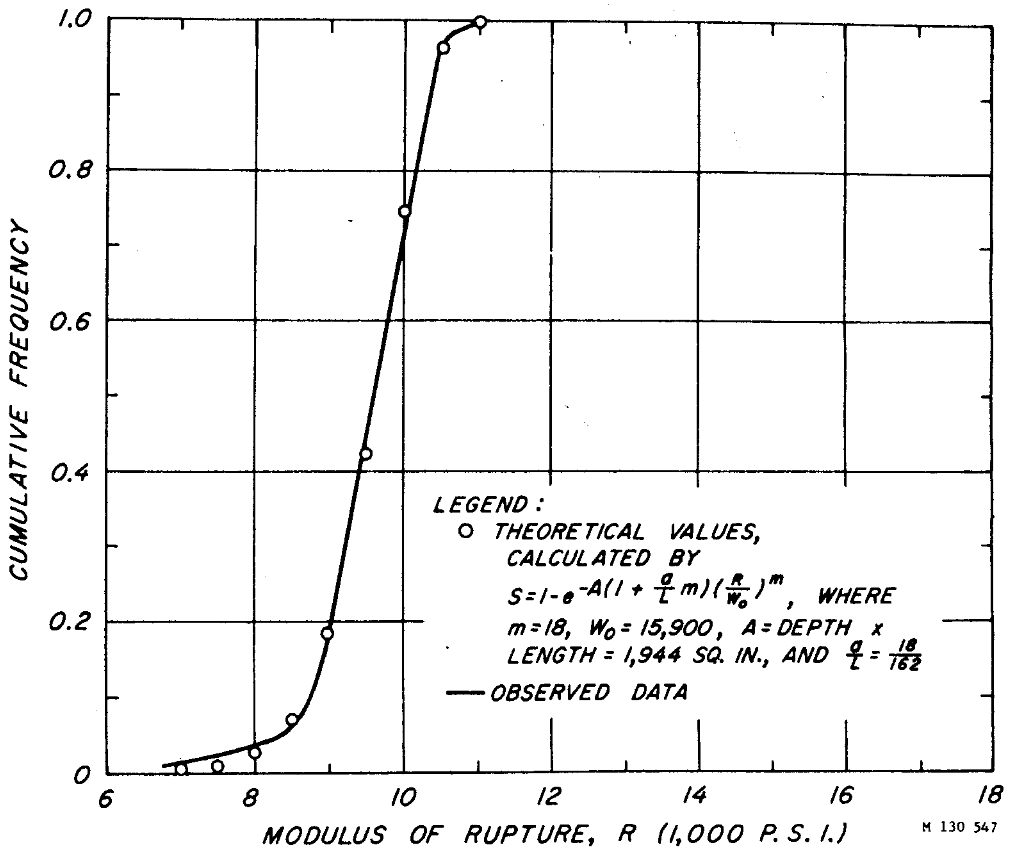

Figure 12.--Cumulative frequency distribution of modulus of rupture of beans 5.2 by 12 by 162 inches, two-point loaded, with 18 inches between load points. Theoretical calculations based on aspect area of beams within span.

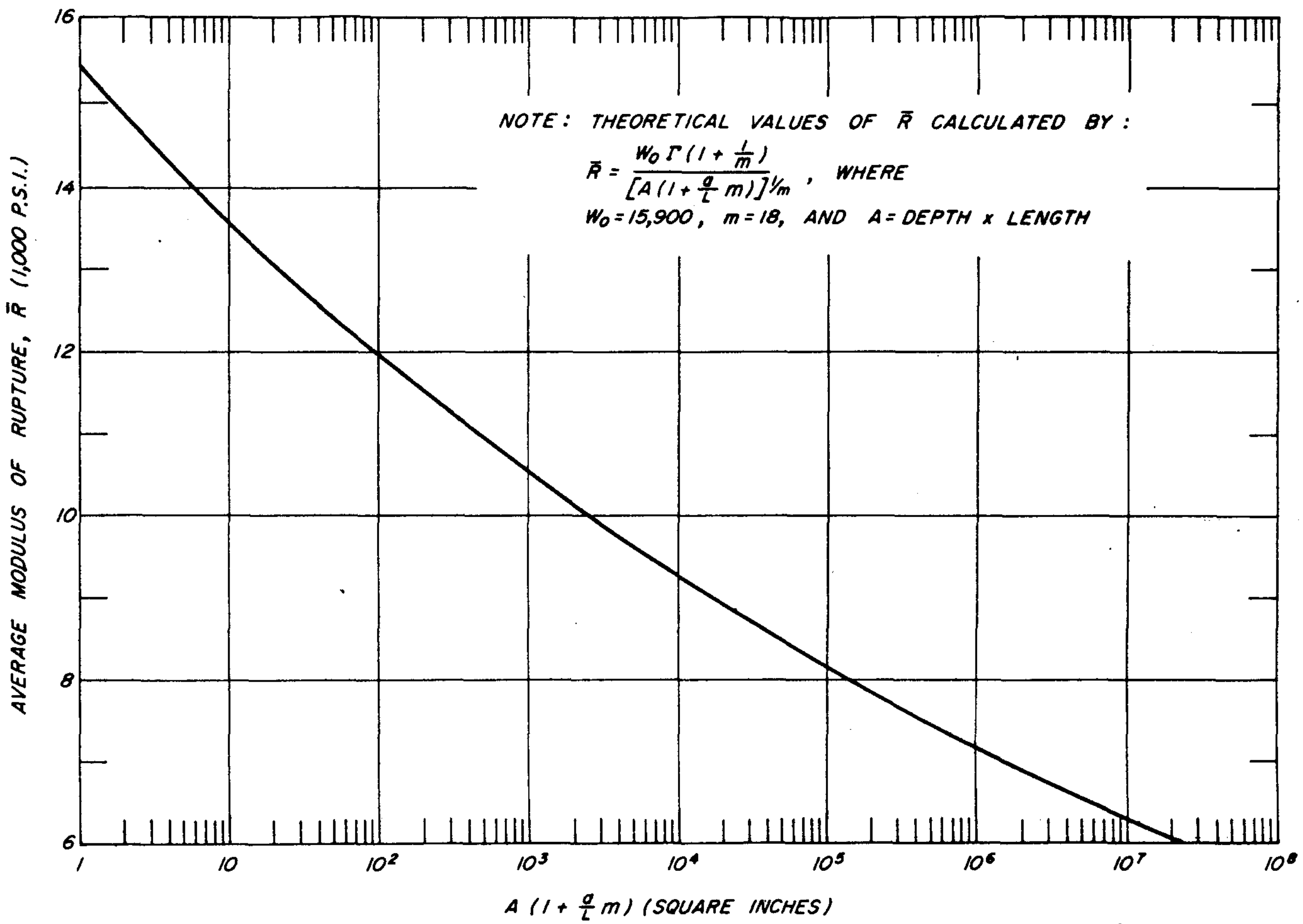

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Figure 13.--Theoretical relationship between average modulus of rupture, $\bar{R}$, of Douglas-fir beams and size and method of loading.

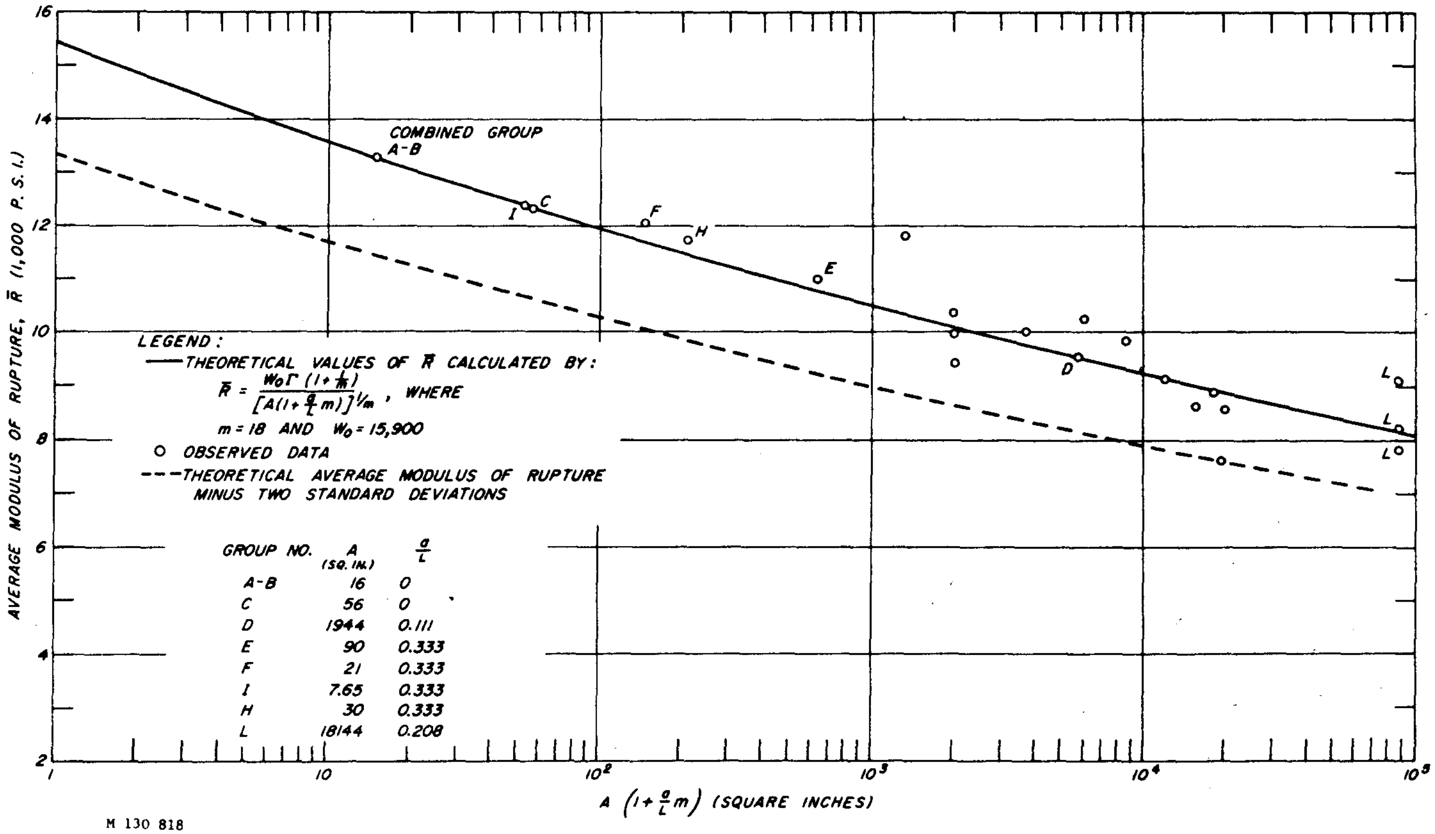

Figure 14.--Comparison between theoretical and actual average modulus of rupture for Douglas-fir beams of different sizes and evaluated under different methods of loading.

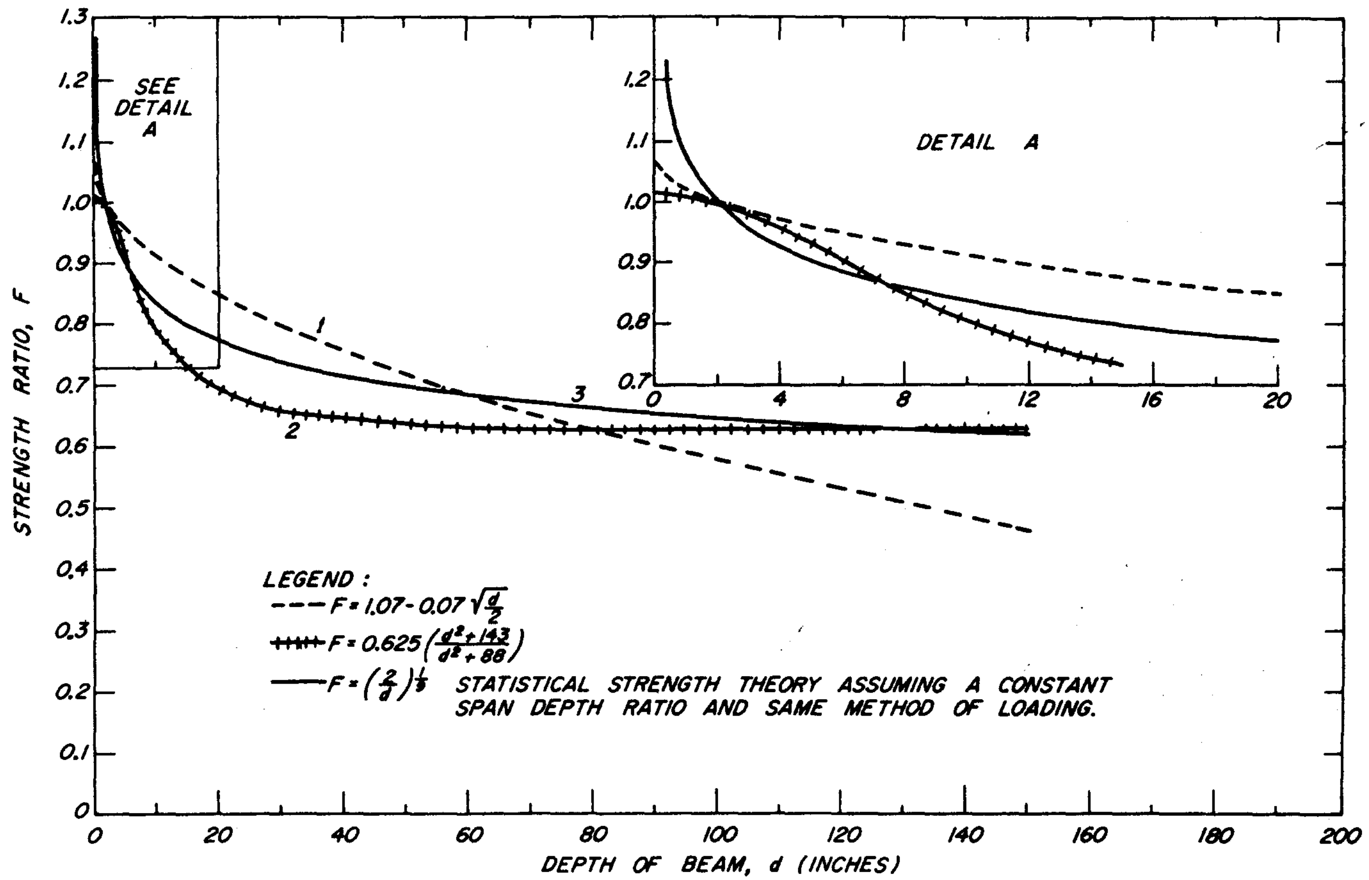

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Figure 15.--Strength ratio, F, of beams having a depth, d, to beams having a depth of 2 inches. To compare curves one and two (based on' older-theories) to the statistical strength of material concept, curve three, a constant span-depth ratio and the same method of loading were assumed in calculating values in the latter theory.