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STRESS DISTRIBUTION IN AN ORTHOTROPIC HALF-PLANE SUBJECTED TO A CONCENTRATED LOAD

ABSTRACT

Mathematical expressions are derived for the stress distribution in wood subjected to a concentrated load. The orthotropic nature of wood was taken into account in the derivation. The analysis was also used to determine the stress distribution in an orthotropic wedge subjected to a concentrated load at its vertex.

STRESS DISTRIBUTION IN AN ORTHOTROPIC HALF-PLANE SUBJECTED TO A CONCENTRATED LOAD

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INTRODUCTION

It is sometimes desirable in a stress-strain analysis to be able to approximate the stress condition existing in the neighborhood of a concentrated load. It is the purpose of this paper to present an exact mathematical solution to the problem as related to an orthotropic half-plane and an orthotropic wedge.

These problems have been solved by other authors[2,3] but results were not presented in a closed useful form.

Conway[4] presents a straightforward solution with the stress distribution formulated as to be easily worked with; however, in his solution he assumes a relationship between the elastic constants, which in effect restricts the solution to a special form of orthotropy. The mathematical procedure in this paper will closely follow that of Conway with the exception of maintaining general orthotropic behavior.

NOTATIONS

x, y	Cartesian coordinates coinciding with natural wood axis.
r, θ	Radial and tangential coordinates, respectively.
ε_x	Extensional strain in the $\underline{x}$ direction.

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[1] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

[2] Benthem, J.P. On the stress distribution in anisotropic infinite wedges. Jour. Appl. Math. XX1(3):189-198, Oct. 1963.

[3] Green, A.E., and Zerna, W. Theoretical elasticity. Oxford, Clarendon Press, 1954.

[4] Conway, H.D. Some problems of orthotropic plane stress, Jour. Appl. Mech. 20(1):72-76, 1953.

ε_y	Extensional strain in the y direction.
ε_{xy}	Shear strain.
σ_x	Normal stress component in the x direction.
σ_y	Normal stress component in the y direction.
σ_{xy}	Shear stress component associated with the x-y plane.
E_x	Modulus of elasticity of natural axis of wood coinciding with x axis.
E_y	Modulus of elasticity of natural axis of wood coinciding with y axis.
G_{xy}	Modulus of rigidity associated with shear deformations in x-y plane.
μ_{yx}	Poisson's ratio of the contraction in the y direction to the extension in the x direction due to a normal tensile stress in x direction.
μ_{xy}	Poisson's ratio of the contraction in the x direction to the extension in y direction due to a normal tensile stress in y direction.
α_1, α_2	Constants dependent upon the material properties.
ϕ	Airy stress function.
P	Concentrated load, per unit width, positive indicating tensile and negative indicating compressive.
Ψ	Half of total wedge angle.

MATHEMATHCAL ANALYSIS

A. Half-Plane Subjected to a Concentrated Load

The problem discussed is assumed to be an orthotropic solid in which the boundaries are assumed to be far removed from the point of application of the concentrated load. In the analysis wood is considered to be orthotropic, and the thickness dimension is considered small so that the problem can be treated as one of plane stress.

The reference axis and load application point are as presented in figure 1.

From the theory of elasticity, the equations of equilibrium are:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \qquad (1)$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} = 0 \qquad (2)$$

and the compatibility equation:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \tag{3}$$

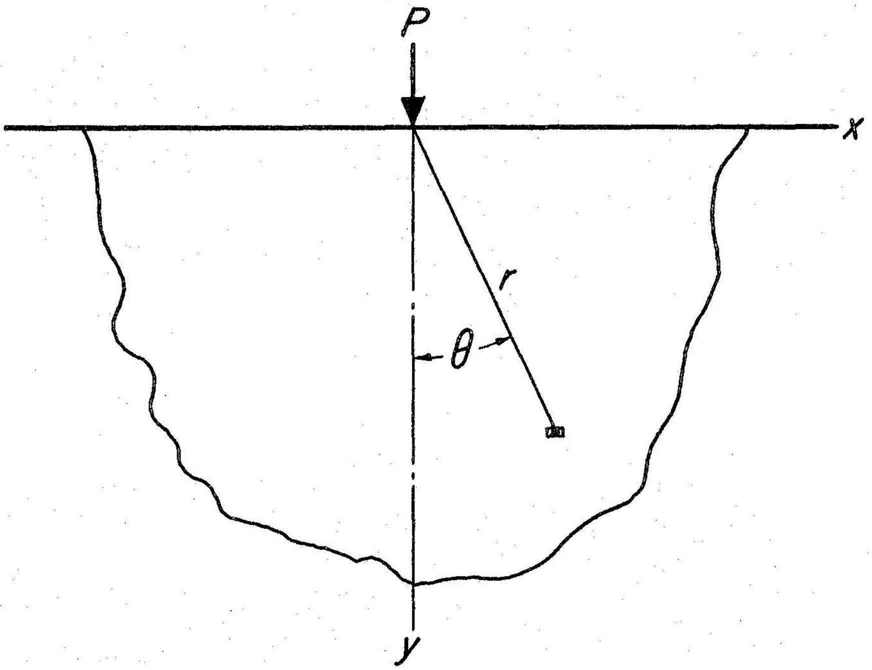

Figure 1.--Half-plane subjected to a normal concentrated load.

For orthotropic materials;

$$\varepsilon_x = \left(\frac{\sigma_x}{E_x} - \mu_{yx} \frac{\sigma_y}{E_y} \right) \tag{4}$$

$$\varepsilon_y = \left(\frac{\sigma_y}{E_y} - \mu_{xy} \frac{\sigma_x}{E_x} \right) \tag{5}$$

$$\varepsilon_{xy} = \frac{\sigma_{xy}}{G_{xy}} \tag{6}$$

Also from the consideration of the strain energy of deformation of an orthotropic material it was found:[5]

$$\frac{\mu_{xy}}{E_x} = \frac{\mu_{yx}}{E_y} \tag{7}$$

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[5] U.S. Forest Products Laboratory. Stress-strain relations in wood and plywood considered as orthotropic materials. FPL Rpt. 1503, 1944.

Therefore equations (4) and (5) may be written:

$$\varepsilon_x = \frac{1}{E_x}(\sigma_x - \mu_{xy}\sigma_y) \qquad (8)$$

$$\varepsilon_y = \frac{1}{E_y}(\sigma_y - \mu_{yx}\sigma_x) \qquad (9)$$

Assuming an Airy stress function $\underline{\phi}$ exists such that:

$$\left.\begin{aligned} \frac{\partial^2\phi}{\partial y^2} &= \sigma_x \\ \frac{\partial^2\phi}{\partial x^2} &= \sigma_y \\ \frac{\partial^2\phi}{\partial x\partial y} &= -\sigma_{xy} \end{aligned}\right\} \qquad (10)$$

It follows by substitution that a differential equation for $\underline{\phi}$ is:

$$\frac{\partial^4\phi}{\partial x^4} + \left(\frac{E_y}{G_{xy}} - \frac{2\mu_{xy}E_y}{E_x}\right)\frac{\partial^4\phi}{\partial x^2\partial y^2} + \frac{E_y}{E_x}\ \frac{\partial^4\phi}{\partial y^4} = 0 \qquad (11)$$

There are two constants which exist such that:

$$\alpha_1\alpha_2 = \frac{E_y}{E_x}$$

$$\alpha_1 + \alpha_2 = \left(\frac{E_y}{G_{xy}} - \frac{2\mu_{xy}E_y}{E_x}\right)$$

Therefore equation (11) may be written:

$$\left(\frac{\partial^2}{\partial x^2} + \alpha_1\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial x^2} + \alpha_2\frac{\partial^2}{\partial y^2}\right)\phi = 0 \qquad (12)$$

The differential equation can be reduced to a simpler form by making the change in variable:

$$y = \sqrt{\alpha_2}\ \eta$$

Therefore equation (12) can be written:

$$\left(\frac{\partial^2}{\partial x^2} + k^2 \frac{\partial^2}{\partial \eta^2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \eta^2}\right)\phi = 0 \tag{13}$$

where:

$$k^2 = \left(\frac{\alpha_1}{\alpha_2}\right) \tag{14}$$

Equation (13) is now in a form similar to that given by Conway[4] and use can be made of the method and stress function ϕ he proposes. The stress function ϕ is in the Fourier integral form:

$$= \int_0^\infty \frac{1}{\beta^2}\left(Ae^{-\beta\eta} + Be^{-\frac{\beta\eta}{k}}\right)\cos\beta x d\beta \tag{15}$$

where A and B are arbitrary constants depending on k. This function yields the stresses:

$$= \frac{\partial^2\phi}{\partial y^2} = \frac{1}{\alpha_2}\frac{\partial^2\phi}{\partial \eta^2} = \frac{1}{\alpha_2}\int_0^\infty \left(Ae^{-\beta\eta} + \frac{B}{k^2}e^{-\frac{\beta\eta}{k}}\right)\cos\beta x d\beta \tag{16}$$

$$= \frac{\partial^2\phi}{\partial x^2} = -\int_0^\infty \left(Ae^{-\beta\eta} + Be^{-\frac{\beta\eta}{k}}\right)\cos\beta x d\beta \tag{17}$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{1}{\sqrt{\alpha_2}}\frac{\partial^2\phi}{\partial x\partial\eta} = -\frac{1}{\sqrt{\alpha_2}}\int_0^\infty \left(Ae^{-\beta\eta} + \frac{B}{k}e^{-\frac{\beta\eta}{k}}\right)\sin\beta x d\beta \tag{18}$$

The normal loading on the boundary, y = 0, can be represented[4] in the form:

$$\lim_{c \to 0} \frac{P}{\pi}\int_0^\infty \frac{\sin\beta c}{\beta c}\cos\beta x d\beta$$

where P is the load per unit width. At y = 0 the boundary conditions are:

$$\sigma_y\Big|_{y=0} = 0 = -\int_0^\infty (A + B)\cos\beta x d\beta = \lim_{c \to 0}\frac{P}{\pi}\int_0^\infty \frac{\sin\beta c}{\beta c}\cos\beta x d\beta \tag{19}$$

$$\sigma_{xy}\Big|_{y=0} = 0 = -\frac{1}{\sqrt{\alpha_2}}\int_0^\infty \left(A + \frac{B}{k}\right)\sin\beta x d\beta \tag{20}$$

since the $\lim\limits_{c \to 0} \frac{\sin\beta c}{\beta c} = 1$, equation (19) can be written:

$$\int_0^\infty \left(\frac{P}{\pi} + A + B\right) \cos\beta x d\beta = 0$$

Consequently:

$$A + B = -\frac{P}{\pi}$$

from equation (20):

$$A + \frac{B}{k} = 0$$

Therefore:

$$A = -\frac{P}{\pi(1 - k)} \qquad (21)$$

$$B = \frac{Pk}{\pi(1 - k)} \qquad (22)$$

The stresses can now be written:

$$\sigma_x = -\frac{P}{\pi(1 - k)\alpha_2}\left\{\int_0^\infty e^{-\beta\eta}\cos\beta x d\beta - \frac{1}{k}\int_0^\infty e^{-\frac{\beta\eta}{k}}\cos\beta x d\beta\right\}$$

$$\sigma_x = -\frac{P}{\pi(1 - k)\alpha_2}\left\{\frac{\eta}{x^2 + \eta^2} - \frac{\eta}{k^2x^2 + \eta^2}\right\}$$

$$\sigma_x = \frac{P(1 + k)x^2\eta}{\pi\alpha_2(x^2 + \eta^2)(k^2x^2 + \eta^2)}$$

Similarly:

$$\sigma_y = \frac{P(1 + k)\eta^3}{\pi(x^2 + \eta^2)(k^2x^2 + \eta^2)}$$

$$\sigma_{xy} = \frac{P(1 + k)x\eta^2}{\pi\sqrt{\alpha_2}(x^2 + \eta^2)(k^2x^2 + \eta^2)}$$

However, since $\eta = \frac{y}{\sqrt{\alpha_2}}$, the stresses can be written in the original coordinate system as:

$$\sigma_x = \frac{\left(\sqrt{\alpha_1} + \sqrt{\alpha_2}\right)Px^2y}{\pi(\alpha_2x^2 + y^2)(\alpha_1x^2 + y^2)} \tag{23}$$

$$\sigma_y = \frac{\left(\sqrt{\alpha_1} + \sqrt{\alpha_2}\right)Py^3}{\pi(\alpha_2x^2 + y^2)(\alpha_1x^2 + y^2)} \tag{24}$$

$$\sigma_{xy} = \frac{\left(\sqrt{\alpha_1} + \sqrt{\alpha_2}\right)Pxy^2}{\pi(\alpha_2x^2 + y^2)(\alpha_1x^2 + y^2)} \tag{25}$$

The stresses may also be expressed in polar form, by making the substitution:

$$y = r\cos\theta$$

$$x = r\sin\theta$$

$$r^2 = x^2 + y^2$$

Therefore:

$$\left.\begin{aligned} \sigma_x &= K\sin^2\theta \\ \sigma_y &= K\cos^2\theta \\ \sigma_{xy} &= K\sin\theta\cos\theta \end{aligned}\right\} \tag{26}$$

where:

$$K = \frac{\left(\sqrt{\alpha_1} + \sqrt{\alpha_2}\right)P\cos\theta}{\pi r\left[\alpha_1\alpha_2\sin^4\theta + (\alpha_1 + \alpha_2)\sin^2\theta\cos^2\theta + \cos^4\theta\right]} \tag{27}$$

The stresses may be written in polar form by means of the transformation equations:6

$$\left.\begin{aligned}\sigma_r &= \sigma_y\cos^2\theta + \sigma_x\sin^2\theta + 2\sigma_{xy}\sin\theta\cos\theta\\ \sigma_\theta &= \sigma_y\sin^2\theta + \sigma_x\cos^2\theta - 2\sigma_{xy}\sin\theta\cos\theta\\ \sigma_{r\theta} &= -\sigma_y\sin\theta\cos\theta + \sigma_x\sin\theta\cos\theta + \sigma_{xy}(\cos^2\theta - \sin^2\theta)\end{aligned}\right\} \tag{28}$$

resulting in:

$$\left.\begin{aligned}\sigma_r &= K\\ \sigma_\theta &= 0\\ \sigma_{r\theta} &= 0\end{aligned}\right\} \tag{29}$$

B. Orthotropic Wedge Subjected to an Axial Concentrated Load

The results of part A are in a form as to be readily adaptable to the solution of an orthotropic wedge subjected to an axial concentrated load, as illustrated in figure 2.

For such a problem, assume the stress distribution in the form:

$$\sigma_r = \frac{A\cos\theta}{r(\alpha_1\sin^2\theta + \cos^2\theta)(\alpha_2\sin^2\theta + \cos^2\theta)} \tag{30}$$

$$\sigma_\theta = 0 \tag{31}$$

$$\sigma_{r\theta} = 0 \tag{32}$$

where the constant A is determined from the condition of vertical equilibrium, or

$$2\int_0^{\Psi} \frac{A\cos\theta \cdot r\cos\theta d\theta}{r(\alpha_1\sin^2\theta + \cos^2\theta)(\alpha_2\sin^2\theta + \cos^2\theta)} = P$$

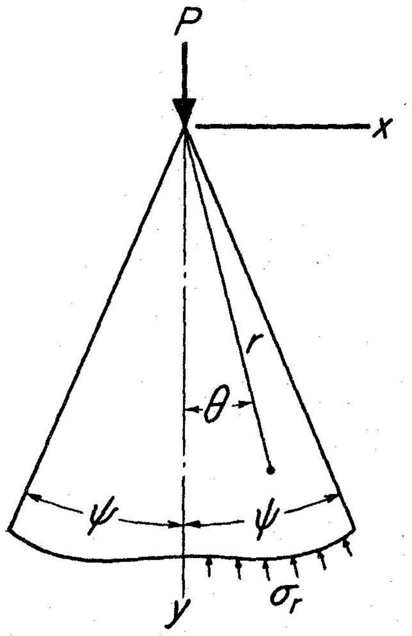

Figure 2.--Orthotropic wedge subjected to an axial concentrated load.

Therefore:

$$A = \frac{P}{2\displaystyle\int_0^{\Psi} \frac{\cos^2\theta d\theta}{(\alpha_1\sin^2\theta + \cos^2\theta)(\alpha_2\sin^2\theta + \cos^2\theta)}}$$

Integrating, it can be found

$$A = \frac{(\alpha_1 - \alpha_2)P}{2\left[\sqrt{\alpha_1}\tan^{-1}(\sqrt{\alpha_1}\tan\Psi) - \sqrt{\alpha_2}\tan^{-1}(\sqrt{\alpha_2}\tan\Psi)\right]}$$

The stresses can now be written:

$$\sigma_r = \frac{(\alpha_1 - \alpha_2)P\cos\theta}{2r(\alpha_1\sin^2\theta + \cos^2\theta)(\alpha_2\sin^2\theta + \cos^2\theta)\left[\sqrt{\alpha_1}\tan^{-1}(\sqrt{\alpha_1}\tan\Psi) - \sqrt{\alpha_2}\tan^{-1}(\sqrt{\alpha_2}\tan\Psi)\right]}$$

$$\sigma_\theta = 0$$

$$\sigma_{r\theta} = 0$$

In the x-y plane, the stresses may be written:

$$\sigma_y = \sigma_r\cos^2\theta$$

$$\sigma_x = \sigma_r\sin^2\theta$$

$$\sigma_{xy} = \sigma_r\sin\theta\cos\theta$$

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