U.S. DEPARTMENT OF AGRICULTURE • FOREST SERVICE
FOREST PRODUCTS LABORATORY • MADISON, WIS.
In Cooperation with the University of Wisconsin

U.S.D.A. FOREST SERVICE
RESEARCH NOTE
FPL-086
Revised NOVEMBER 1970

MINIMUM WEIGHT STRUCTURAL SANDWICH

This Report is One of a Series
Issued in Cooperation with the
MIL-HDBK-23 WORKING GROUP ON
STRUCTURAL SANDWICH COMPOSITES
FOR AEROSPACE VEHICLES
of the Departments of the
AIR FORCE, NAVY, AND COMMERCE

Abstract

This note presents theoretical analyses for determination of dimensions of structural sandwich of minimum weight that will have certain stiffness and load-carrying capabilities. Included is a brief discussion of the resultant minimum weight configurations.

MINIMUM WEIGHT STRUCTURAL SANDWICH[1]

By

EDWARD W. KUENZI, Engineer

Forest Products Laboratory,[2] Forest Service
U.S. Department of Agriculture

Introduction

The concept of sandwich construction combining thin, strong facings on lightweight, thick cores immediately suggests possibilities of deriving constructions so proportioned that minimum weight for a given stiffness or loading capability is achieved. It is important to realize that the minimum weight construction derived may not be practical because of unusually thin facings which are not available, or some other detail such as an unusually lightweight core of great thickness. Since it is theoretically possible to arrive at impractical designs, various minimum weight analyses should be used with caution for comparing sandwich with other constructions unless the sandwich proportions are examined. Analyses of the efficiency of panels of various sandwich constructions of certain materials have been reported.[3,4] This note presents some general analyses of minimum weight sandwich considering stiffness, edge load capacity, and bending moment capacity.

---

[1] Revision of a Note of the same title and designation, published in October 1965. Research was performed for the Military Handbook 23 Working Group by the Forest Products Laboratory under U.S. Air Force Contract No. 33(616)70-M-5000. Results are preliminary and may be revised as additional data become available.

[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

[3] Johnson, Aldie E., Jr., and Semonian, Joseph W. A study of the efficiency of high-strength, steel, cellular-core sandwich plates in compression. National Advisory Committee for Aeronautics Tech. Note 3751, Sept. 1956.

[4] Kaechele, L. E. Minimum-weight design of sandwich panels. Rand Corp. Rpt. RM-1895, March 1957.

Bending Stiffness

Since the primary purpose of structural sandwich is to provide stiffness (hence low deflection under transverse load and high resistance to buckling under edgewise load), the analysis of minimum weight sandwich to provide a specified bending stiffness is considered first.

Sandwich bending stiffness per unit width can be derived by elementary mechanics and is given by the following formula for sandwich with thin facings and a core of negligible bending stiffness:

$$D = \frac{\frac{E_1 t_1}{\lambda_1} \frac{E_2 t_2}{\lambda_2} h^2}{\frac{E_1 t_1}{\lambda_1} + \frac{E_2 t_2}{\lambda_2}} \tag{1}$$

where D is bending stiffness; subscripts 1 and 2 denote facings 1 and 2; E is modulus of elasticity; λ is one minus the product of two Poisson's ratios; t is thickness; and h is distance between facing centroids.

After setting

$$\beta = \frac{E_2 t_2 \lambda_1}{E_1 t_1 \lambda_2} \tag{2}$$

formula (1) can be rewritten as

$$D = \frac{E_1 t_1}{\lambda_1} h^2 \frac{\beta}{1 + \beta} \tag{3}$$

The weight of a sandwich is given by the formula

(4)

where w is density (p.c.i.); W_B is total weight of bond (adhesive or brace) between facings and core (p.s.i.); W is sandwich weight (p.s.i.), and

t_c is core thickness. The core thickness can be defined in terms of h and the facing thicknesses by

$$t_c = h - \frac{t_1+t_2}{2}$$

And after defining new parameters

$$\phi_1 = w_1 - \frac{w_c}{2}$$

$$\phi_2 = w_2 - \frac{w_c}{2}$$

and substitution into (4) the resultant weight expression, after assuming bond weight, W_B, is the same for all sandwich of the type considered; it can then be written as

$$(W-W_B) = \phi_1 t_1 + \phi_2 t_2 + w_c h \tag{5}$$

Solution of (2) for t_2 and (3) for t_1 and substituting into (5) results in:

$$(W-W_B) = \frac{D\lambda_1}{E_1}\left(\phi_1 + \beta\frac{E_1\lambda_2}{E_2\lambda_1}\phi_2\right)^{\frac{1+\beta}{\beta}} h^{-2} + w_c h \tag{6}$$

Expression (6) is to be minimized with respect to h and ß. Minimizing (6) with respect to h results in

$$h^3 = \frac{2D\lambda_1}{E_1 w_c}\left(\phi_1 + \beta\frac{E_1\lambda_2}{E_2\lambda_1}\phi_2\right)^{\frac{1+\beta}{\beta}} \tag{7}$$

The configuration of this minimum weight sandwich can be examined by substituting (3) for D in formula (7) and (2) for ß in formula (7) to produce the following expression:

$$w_c h = 2(\phi_1 t_1 + \phi_2 t_2)$$

The left–hand portion of this is approximately equal to the core weight of the sandwich and the right–hand portion approximately double the facing weight. Thus for a minimum weight sandwich of a specified bending stiffness the core weight must be approximately two–thirds the weight of the sandwich minus bond weight.

Values of facing thicknesses can be found by substituting (3) for D in formula (7) to give

$$t_1 = \frac{w_c h}{2\left(\phi_1 + \beta \frac{E_1 \lambda_2}{E_2 \lambda_1} \phi_2\right)} \qquad (8)$$

Minimization of (6) with respect to ß results in

$$\beta^2 = \frac{E_2 \lambda_1 \phi_1}{E_1 \lambda_2 \phi_2} \qquad (9)$$

and substituting (9) into (2) results in

$$t_2 = t_1 \sqrt{\frac{\phi_1 E_1 \lambda_2}{\phi_2 E_2 \lambda_1}} \qquad (10)$$

Additional insight concerning optimum sandwich can be had by proportioning facings to minimize stiffness. This can be done by rewriting formula (1) as

$$\frac{D}{h^2} = \frac{\alpha(\eta - \alpha)}{\eta} \qquad (11)$$

where

$$\alpha = \frac{E_1 t_1}{\lambda_1}$$

and

$$\eta = \frac{E_1 t_1}{\lambda_1} + \frac{E_2 t_2}{\lambda_2}$$

Since a certain amount of facing is needed to carry loads (bending or compression) η can be considered to be constant and maximizing (11) with respect to α results in $\alpha = \eta/2$ which finally produces the relationship that the facings shall have equal extensional stiffnesses, i.e.

$$\frac{E_1 t_1}{\lambda_1} = \frac{E_2 t_2}{\lambda_2} \tag{12}$$

and substitution of (12) into (10) results in

$$\phi_1 t_1 = \phi_2 t_2 \tag{13}$$

which states that the weights of the facings shall be nearly the same.

The foregoing has not considered high stresses that might have to be carried by very thin facings. Because of stress limitations, availability of facings in proper thicknesses, and availability of cores of low density, it will often be found that the minimum weight sandwich cannot be realized. It is also important to review other discussions of minimum weight sandwich to be sure that inherently impossible combinations of unusually thin facings on lightweight cores are not being examined.

Bending Moment Capacity

The bending moment resistance of a sandwich with thin, equal facings on a core of negligible bending stiffness is given by the formula

$$M = Fth \qquad (14)$$

where $\underline{M}$ is bending moment per unit width; $\underline{F}$ is design facing stress, $\underline{t}$ is facing thickness; and $\underline{h}$ is distance between facing centroids. Following the same procedure as for bending stiffness, formula (14) is solved for $\underline{t}$ and substitution of this in the weight formula $(W - W_B) = 2wt + w_c h$ results in

$$\left(W - W_B\right) = 2w \frac{M}{F} h^{-1} + w_c h \qquad (15)$$

Minimizing (15) with respect to $\underline{h}$ results in

$$h^2 = 2 \frac{w}{w_c} \frac{M}{F} \qquad (16)$$

The configuration of this minimum weight sandwich can be examined by substituting (14) for $\underline{M}$ in formula (16) and obtaining, finally

$$\frac{t}{h} = \frac{w_c}{2w} \qquad (17)$$

and further substitution of (17) into the weight formula leads to

$$W_c = \frac{1}{2}\left(W - W_B\right) \tag{18}$$

Thus the core weight for a minimum weight sandwich of specified moment capacity must be one–half the weight of the sandwich minus bond weight.

The moment capacity was based on a facing stress, F, which may be an allowable stress or failing stress, etc. If the possibility of local wrinkling or dimpling of sandwich facings exists, then moment capacity should be based on that stress at which wrinkling or dimpling of facings occurs.

The wrinkling stress of sandwich facings is dependent upon facing and core properties. Since core properties can be related to core density, wrinkling stress can also be related to density. The wrinkling stress of sandwich facings is often given by the formula

$$F_w = k\left(E E_c G_c\right)^{1/3} \tag{19}$$

where F_w is wrinkling stress of facings; k is a theoretical or empirical buckling coefficient; E is effective elastic modulus of facing: E_c is core modulus of elasticity (normal to facings): G_c is core shear modulus. If it is assumed that core elastic properties are related to facing properties in proportion to densities as follows

$$E_c = k_1 \frac{w_c}{w} E$$

$$G_c = k_2 \frac{w_c}{w} E$$

then formula (19) can be written as

$$F = KE\left(\frac{w_c}{w}\right)^{2/3} \tag{20}$$

where $K = k\left(k_1 k_2\right)^{1/3}$

Since F (formula 20) is not dependent upon t or h, the dimensions of the minimum weight sandwich for face wrinkling can be determined directly from the previous analysis.

Substituting (20) into (16) yields

$$h^2 = \frac{2M}{KE\left(\frac{w_c}{w}\right)^{5/3}} \tag{21}$$

which is dependent upon relative density of core and facing materials, but as before

$$\frac{t}{h} = \frac{w_c}{2w}$$

which is the same as (17). Therefore, the minimum weight sandwich for which bending moment resistance based on wrinkling of the compression facings must have a core weight of one-half the sandwich weight minus bond weight.

The dimpling stress of sandwich facings on honeycomb or corrugated core is dependent upon facing properties and unsupported width of facing. The dimpling stress is given by the formula

$$F_D = kE\frac{t^2}{s^2} \tag{22}$$

where F_D is dimpling stress of facings; k is a theoretical or empirical buckling coefficient; E is effective elastic modulus of facing; t is facing thickness: and s is honeycomb core cell size or spacing between points of corrugated core supports for the facings. If it is assumed that the core density is related to the facing density and is inversely proportional to s, then

$$s = k_1 \frac{w}{w_c}$$

and (22) becomes

$$F_D = KEt^2\left(\frac{w_c}{w}\right)^2 \tag{23}$$

where $K = \frac{k}{k_1^2}$

Proceeding as before and solving formula (14) for h rather than t, thus avoiding some cube roots after substituting (23) for F, results in

$$h = \frac{Mt^{-3}}{KE\left(\frac{w_c}{w}\right)^2} \tag{24}$$

Substituting (24) into the weight equation $(W - W_B) = 2wt + w_c h$ results in

$$\left(W - W_B\right) = 2wt + \frac{w_c Mt^{-3}}{KE\left(\frac{w_c}{w}\right)^2} \tag{25}$$

and minimizing this expression with respect to t finally yields

$$t^4 = \frac{3M}{2KE\left(\frac{w_c}{w}\right)} \tag{26}$$

From equations (24) and (26) it is found that

$$\frac{t}{h} = \frac{3w_c}{2w} \tag{27}$$

and substituting (27) into the weight equation finally results in the core weight

$$W_c = \frac{1}{4}(W - W_B) \qquad (28)$$

Buckling Under Compressive Edge Load

The edge load capacity of a sandwich panel, precluding local facing failures by wrinkling, dimpling, or facing compression failure, is dependent on the buckling of the entire sandwich. This buckling is determined not only by the sandwich bending stiffness, D, but also by the shear stiffness.

The buckling load, per unit width, of a simply supported flat sandwich panel with isotropic facings and core, and having a length not less than its loaded width is given by the formula[5]

$$N = K \frac{\pi^2}{b^2} D \qquad (29)$$

where $D = \frac{Eth^2}{2\lambda}$

$$K = \frac{4}{(1 + V)^2}$$

$$V = \frac{\pi^2 Eth}{2\lambda b^2 G_c}$$

$$\lambda = 1 - \mu^2$$

[5] Ericksen, Wilhelm S., and March, H. W. Compressive buckling of sandwich panels having dissimilar facings of unequal thickness. Forest Prod. Lab. Rept. 1583-B, Rev. 1958.

and N is buckling load per unit panel width; b is panel width (loaded edge); E is facing elastic modulus: t is facing thickness; h is distance between facing centroids; G_c is core shear modulus; and μ is facing Poisson's ratio. After substituting values for D, K, and V in equation (29) and defining

$$A = \frac{2\pi^2 E}{\lambda b^2}$$

equation (29) becomes

$$N = \frac{Ath^2}{\left(1 + \frac{A}{4G_c} th\right)^2} \tag{30}$$

The sandwich weight equation is

$$(W - W_B) = 2wt + w_c h \tag{31}$$

Elimination of one of the variables, t or h, by solving (30) and substituting into (31) is not as easy as in previous examples. However the minimum of (31) subjected the constraint imposed by (30) can be found by a method used by Lagrange by solving the following:

$$\frac{\partial N}{\partial t} + L \frac{\partial W}{\partial t} = 0$$

$$\frac{\partial N}{\partial h} + L \frac{\partial W}{\partial h} = 0 \tag{32}$$

where $\underline{L}$ is a Lagrangian multiplier. After performing the partial differentiation indicated by (32) and reducing, the two expressions of (32) become:

$$Ath\left(1 - \frac{Ath}{4G_c}\right) + 2w\frac{t}{h}\left(1 + \frac{Ath}{4G_c}\right)^3 L = 0$$

$$Ath + \frac{1}{2}w_c\left(1 + \frac{Ath}{4G_c}\right)^3 L = 0 \tag{33}$$

And after noting that $= \frac{Ath}{4G_c}$ expressions (33) can be written as:

$$Ath(1 - V) + 2w\frac{t}{h}(1 + V)^3 L = 0$$

$$- Ath(1 - V) - \frac{1}{2}w_c(1 - V)(1 + V)^3 L = 0 \tag{34}$$

Adding the expressions of (34) finally results in

$$\frac{t}{h} = \frac{w_c}{4w}(1 - V) \tag{35}$$

Solving (35) for $\underline{t}$ and substituting into (30) and solving for $\underline{h}$ gives

$$h^3 = \frac{N\lambda b^2(1 + V)^2}{2\pi^2E(1 - V)\frac{w_c}{4w}} \tag{36}$$

Equation (36) can be used to find $\underline{h}$ by first assuming V = 0 to obtain in a minimum $\underline{h}$ and then using a finite $\underline{V}$ after assuming a core shear modulus, $\underline{G_c}$, and determining $\underline{t}$ from (35).

From the weight equation

$$\frac{(W - W_B)}{wh} = 2\frac{t}{h} + \frac{w_c}{w} \tag{37}$$

and substitution of (35) into (37) results finally in

$$\frac{(W - W_B)}{wh} = \frac{w_c}{w}\left(\frac{3}{2} - \frac{V}{2}\right) \tag{38}$$

or

$$(W - W_B) = w_c h\left(\frac{3}{2} - \frac{V}{2}\right)$$

and finally if $W_c = w_c h$

$$\frac{W_c}{(W - W_B)} = \frac{2}{3 - V} \tag{39}$$

Thus if the core shear modulus is large and V = 0, the core weight is two–thirds of the sandwich weight minus bond weight. This also was the result obtained when prescribed bending stiffness was analyzed and was to be expected since for V = 0 buckling depends on bending stiffness (see (29)). The effect of a V ≠ 0 reduces the core weight relative to the sandwich weight.

Buckling of Sandwich Cylinders Under Axial Compressive Load

The axial compressive load capacity of a circular cylinder with walls of sand–wich construction, precluding local facing failures by wrinkling, dimpling, or facing compression failure, is dependent on the buckling of the sandwich walls.

The buckling load, per unit circumference, of a sandwich walled cylinder in axial compression is given by the formula[6,7]

$$N = KEt\frac{h}{r} \tag{40}$$

where $K = k_1(1 - k_2V)$ for $V < \frac{1}{2}$ (approximately) and $V = \frac{Et}{rG_c}$ (approximately);

$\underline{N}$ is buckling load per unit cylinder circumference: $\underline{E}$ is elastic modulus of facing: $\underline{t}$ is facing thickness; $\underline{h}$ is distance between facing centroids: $\underline{r}$ is mean radius of cylinder; $\underline{k_1}$ is a coefficient dependent upon whether buckling is governed by small or large deflection theory;[6,7] $\underline{k_2}$ is a coefficient depending upon whether isotropic or orthotropic core is used and also upon small or large deflection theory: and $\underline{G_c}$ is core shear modulus associated with shear distortion axially.

Substituting values of $\underline{K}$ and $\underline{V}$ into (40) yields

$$N = k_1\left(1 - k_2\frac{Et}{rG_c}\right)Et\frac{h}{r} \tag{41}$$

and solving for $\underline{h}$ results in

$$h = \frac{rN}{k_1Et\left(1 - k_2\frac{Et}{rG_c}\right)} \tag{42}$$

---

[6]March, H. W., and Kuenzi, Edward W. Buckling of cylinders of sandwich construction in axial compression. Forest Prod. Lab. Rpt. 1830. Rev. Dec.1957.

[7]Zahn, John J., and Kuenzi, Edward W. Classical buckling of cylinders of sandwich construction in axial compression--orthotropiccores. U.S. Forest Serv. Res. Note FPL-018. Nov. 1963. Forest Prod. Lab., Madison, Wis.

Substitution of (42) into the sandwich weight equation $(W - W_B) = 2wt + w_c h$ yields

$$(W - W_B) = 2wt + \frac{rNw_c}{k_1 Et\left(1 - k_2 \frac{Et}{rG_c}\right)} \tag{43}$$

and minimizing (43) with respect to t yields

$$t^2 = \frac{rNw_c\left(1 - 2k_2 V\right)}{2wk_1 E\left(1 - k_2 V\right)^2} \tag{44}$$

From (42) and (44) it can be shown that

$$\frac{t}{h} = \frac{w_c\left(1 - 2k_2 V\right)}{2w\left(1 - k_2 V\right)} \tag{45}$$

Rewriting the weight equation to give

$$\frac{(W - W_B)}{w_c h} = \frac{2w}{w_c}\frac{t}{h} + 1 \tag{46}$$

and substituting (45) into (46) finally results in

$$\frac{w_c h}{(W - W_B)} = \frac{W_c}{(W - W_B)} = \frac{1 - k_2 V}{2 - 3k_2 V} \tag{47}$$

Thus, the core weight, Wc, is determined to be about one–half the sandwich weight for sandwich proportioned to give minimum weight for a given cylinder buckling load. This was to be expected because for V = 0 the buckling load is dependent upon the product th as when prescribed moment resistance was analyzed for which $W_c = \frac{1}{2}(W - W_B)$.