U.S. Forest Service Research Note

UNITED STATES DEPARTMENT OF AGRICULTURE . FOREST SERVICE · FOREST PRODUCTS LABORATORY · MADISON, WIS.

In Cooperation with the University of Wisconsin

STRESSES WITHIN CURVED LAMINATED BEAMS OF DOUGLAS-FIR

NOVEMBER 1963

FPL-020

STRESSES WITHIN CURVED LAMINATED BEAMS OF DOUGLAS-FIR

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Synopsis

The mathematical theory of elasticity is applied to a curved beam made of an orthotropic material, the natural axes of the material being curved with the beam. Curves are given for the determination of the tangential stresses on the concave and convex surfaces of a Douglas-fir beam. The approximate equation of U.S. Department of Agriculture Technical Bulletin 691 for the radial stresses is verified.

General

Curved laminated beams of Douglas-fir and southern yellow pine are rapidly coming into general use as roof beams in many modern buildings. The stresses within these beams have not been determined in a rigorous manner. It seems practical to do so now. The cross section of the beam will be assumed to be rectangular so that a two-dimensional stress system can be employed. Polar coordinates with the pole at the center of the curvature (fig. 1) will be used to simplify the problem. Three strains are given in terms of the two displacements, u in the radial direction and v in the tangential direction by:[2]

$$\epsilon_r = \frac{\partial u}{\partial r} \tag{1}$$

$$\epsilon_\theta = \frac{u}{1} + \frac{1}{r} \frac{\partial v}{\partial \theta} \tag{2}$$

$$\epsilon_{r\theta} = \frac{1}{r} \frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r} \tag{3}$$

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[1] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.
[2] Timoshenko, S. and Goodier, J. Theory of Elasticity. p. 305. McGraw-Hill. 1951.

Two displacements, u and v, give rise to three strains so that these strains are not independent, Elimination of the displacements from equations (1), (2), and (3) yields:

$$\frac{\partial^2 \epsilon_\theta}{\partial r^2} + \frac{2}{r}\frac{\partial \epsilon_\theta}{\partial r} - \frac{1}{r}\frac{\partial^2 \epsilon_{r\theta}}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial \epsilon_{r\theta}}{\partial \theta} - \frac{1}{r}\frac{\partial \epsilon_r}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \epsilon_r}{\partial \theta^2} = 0 \tag{4}$$

Equation (4) is the compatibility equation in polar coordinates in terms of the strains.

The strains are given in terms of the stresses by: [3]

$$\epsilon_\theta = \alpha\sigma_\theta - \beta\sigma_r \tag{5}$$

$$\epsilon_r = \beta\sigma_\theta + \gamma\sigma_r \tag{6}$$

$$\epsilon_{r\theta} = \frac{1}{G_{r\theta}}\sigma_{r\theta} \tag{7}$$

where

$$\alpha = \frac{1}{E_\theta}(1 - \mu_{\theta z}\mu_{z\theta}),\ \beta = \frac{1}{E_\theta}(\mu_{\theta z}\mu_{zr} + \mu_{\theta r}),\ \gamma = \frac{1}{E_r}(1 - \mu_{zr}\mu_{rz}) \tag{8}$$

and E_θ and E_r are the Young's moduli in the tangential and radial directions of the beam, and the μ values are the Poisson's ratios, taken positive, of the strains in the direction of the second subscript to that in the direction of the first subscript due to a stress in the direction of the first subscript. $G_{r\theta}$ is the modulus of rigidity associated with the radial and tangential directions and σ and ϵ are the stress and strain associated with the directions indicated by their subscripts.

[3] March, H. W. and Smith, C. B. Flexural Rigidity of a Rectangular Strip of Sandwich Construction. Forest Products Lab, Report No. 1505. Revised 1955.

The equations of equilibrium in polar coordinates, neglecting body forces, are

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \tag{9}$$

and

$$\frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} = 0 \tag{10}$$

These equations are satisfied by taking a stress function $\underline{\phi}$ such that (Timoshenko and Goodier, page 56)[2]

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} \tag{11}$$

$$\sigma_r = \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \tag{12}$$

$$\sigma_{r\theta} = \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} - \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} \tag{13}$$

This can be verified by substituting (11), (12), and (13) in (9) and (10).

The strains are put in terms of thestress function by substituting (11), (12), and (13) in (5), (6), and (7). Thus:

$$\epsilon_\theta = \alpha \frac{\partial^2 \phi}{\partial r^2} - \frac{\beta}{r} \frac{\partial \phi}{\partial r} - \frac{\beta}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \tag{14}$$

$$\epsilon_r = -\beta \frac{\partial^2 \phi}{\partial r^2} + \frac{\gamma}{r} \frac{\partial \phi}{\partial r} + \frac{\gamma}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} \tag{15}$$

$$\epsilon_{r\theta} = \frac{1}{G_{r\theta}} \frac{1}{r^2} \frac{\partial \phi}{\partial \theta} - \frac{1}{G_{r\theta}} \frac{1}{r} \frac{\partial^2 \phi}{\partial r \partial \theta} \tag{16}$$

We substitute these values of strain in the compatibility equation (4), taking into account the polar symmetry of the problem; that is, all strains and stresses are independent of $\underline{\theta}$. We obtain the ordinary differential equation: (Note ($\sigma_{r\theta} = e_{r\theta}$ = 0 for polar symmetry.)

$$\frac{d^4\phi}{dr^4} + \frac{2}{r}\frac{d^3\phi}{dr^3} - \frac{\gamma}{\alpha}\frac{1}{r^2}\frac{d^2\phi}{dr^2} + \frac{\gamma}{\alpha}\frac{1}{r^3}\frac{d\phi}{dr} = 0 \tag{17}$$

The problem is solved by finding a function $\underline{\phi}$ of $\underline{r}$ that satisfies equation (17) and the boundary conditions:

1. $$\sigma_r = 0 \text{ for } r = a \text{ and } r = b \tag{18}$$

2. $$\int_a^b \sigma_\theta \, dr = 0 \qquad \int_a^b \sigma_\theta r\,dr = -M \tag{19}$$

3. $$\sigma_{r\theta} = 0 \quad \text{at all boundaries} \tag{20}$$

Condition 1 requires that the convex and concave boundaries are free from normal stresses. Condition 2 requires that the normal stress at the ends give rise to the couple $\underline{M}$ only. Condition 3 requires that there are no tangential forces applied at the boundaries.

If the material is isotropic, $\gamma = \alpha$ and equation (17) reduces to equation (41) of Timoshenko and Goodier, page 58.[2]

Equation (17) is multiplied by $\underline{r}^4$ to obtain:

$$r^4\frac{d^4\phi}{dr^4} + 2r^3\frac{d^3\phi}{dr^3} - \frac{\gamma}{\alpha} r^2\frac{d^2\phi}{dr^2} + \frac{\gamma}{\alpha} r\frac{d\phi}{dr} = 0 \tag{21}$$

This is a homogeneous linear differential equation.

Let $\phi = r^m$. Upon the substitution, equation (21) becomes

$$m(m-1)(m-2)(m-3)\,r^m + 2m(m-1)(m-2)\,r^m - \frac{\gamma}{\alpha}m(m-1)\,r^m + \frac{\gamma}{\alpha}mr^m = 0 \tag{22}$$

Dividing by r^m, carrying out the multiplication and collecting terms,

$$m^4 - 4m^3 + \left(5 - \frac{\gamma}{\alpha}\right) m^2 - 2\left(1 - \frac{\gamma}{\alpha}\right) m = 0 \tag{23}$$

Thus the solution of (21) and therefore (17) is:

$$\phi = C_0 r^{m_0} + C_1 r^{m_1} + C_2 r^{m_2} + C_3 r^{m_3} \tag{24}$$

where C_0, C_1, C_2, and C_3, are constants of integration and m_0, m_1, m_2, and m_3, are the four roots of equation (23), they are:

$$\begin{aligned} m_0 &= 0 \\ m_1 &= 2 \\ m_2 &= 1 + \sqrt{\frac{\gamma}{\alpha}} \\ m_3 &= 1 - \sqrt{\frac{\gamma}{\alpha}} \end{aligned} \tag{25}$$

From equations (8)

$$\frac{\gamma}{\alpha} = \frac{E_\theta (1 - \mu_{zr}\mu_{rz})}{E_r (1 - \mu_{\theta z}\mu_{z\theta})} \tag{26}$$

To obtain useful values of $\frac{\gamma}{\alpha}$, we take the laminated beam to be made of plainsawed boards and use the values given for Douglas-fir at 12.9 percent moisture content in table 13 of the Wood Handbook.[4]

$$\frac{\gamma}{\alpha} = \frac{E_L (1 - \mu_{TR}\mu_{RT})}{E_R (1 - \mu_{LT}\mu_{TL})} = 12.7264 \tag{27}$$

[4] U.S. Forest Products Laboratory. Wood Handbook. U.S. Department of Agriculture. Agriculture Handbook 72. 528 pp. 1955.

The roots (25) become:

$$m_0 = 0$$
$$m_1 = 2$$
$$m_2 = 4.567 \qquad (28)$$
$$m_3 = 2.567$$

The radial stress is given by equation (12). The second term of the righthand member of this equation is zero because of axial symmetry. Thus:

$$\sigma_r = \frac{1}{r} \frac{\partial \phi}{\partial r} \qquad (29)$$

Substituting equation (24) in (29), remembering that $m_0 = 0$.

$$\sigma_r = \frac{1}{r}\left[C_1 m_1 r^{m_1 - 1} + C_2 m_2 r^{m_2 - 1} + C_3 m_3 r^{m_3 - 1}\right]$$

or

$$\sigma_r = C_1 m_1 r^{m_1 - 2} + C_2 m_2 r^{m_2 - 2} + C_3 m_3 r^{m_3 - 2} \qquad (30)$$

Two equations are obtained from (30) by substituting in it the boundary conditions expressed by (18):

$$C_1 m_1 b^{m_1 - 2} + C_2 m_2 b^{m_2 - 2} + C_3 m_3 b^{m_3 - 2} = 0 \qquad (31)$$

$$C_1 m_1 a^{m_1 - 2} + C_2 m_2 a^{m_1 - 2} + C_3 m_3 a^{m_3 - 2} = 0 \qquad (32)$$

A third equation may be obtained from the second boundary condition, equations (19). Using equation (11) they may be written

$$\int_a^b \frac{\partial^2 \phi}{\partial r^2} \, dr = 0 \qquad \textbf{(33)}$$

$$\int_a^b \frac{\partial^2 \phi}{\partial r^2} \, r \, dr = -M \tag{34}$$

Performing the integration indicated in equation (33)

$$\left| \frac{\partial \phi}{\partial r} \right|_a^b = 0 \tag{35}$$

Using equation (24):

$$C_1 m_1 b^{m_1 - 1} + C_2 m_2 b^{m_2 - 1} + C_3 m_3 b^{m_3 - 1}$$
$$- C_1 m_1 a^{m_1 - 1} - C_2 m_2 a^{m_2 - 1} - C_3 m_3 a^{m_3 - 1} = 0 \tag{36}$$

Equation (36) is in agreement with equation (31) and (32). The third boundary condition, equation (20), is satisfied everywhere because of polar symmetry. Using equation (24), equation (34) becomes:

$$C_1 (m_1 - 1)(b^{m_1} - a^{m_1}) + C_2 (m_2 - 1)(b^{m_2} - a^{m_2})$$
$$+ C_3 (m_3 - 1)(b^{m_3} - a^{m_3}) = -M \tag{37}$$

multiplying (31) by $\underline{b^2}$ and (32) by $\underline{a^2}$ they become:

$$C_1 m_1 b^{m_1} + C_2 m_2 b^{m_2} + C_3 m_3 b^{m_3} = 0 \tag{38}$$

$$C_1 m_1 a^{m_1} + C_2 m_2 a^{m_2} + C_3 m_c a^{m_3} = 0 \tag{39}$$

In the light of equations (36) and (39) equation (37) becomes:

$$C_1 (b^{m_1} - a^{m_1}) + C_2 (b^{m_2} - a^{m_2}) + C_3 (b^{m_3} - a^{m_3}) = M \tag{40}$$

Equations (38), (39), and (40) form a simultaneous set from which the values of $\underline{C_1}$, $\underline{C_2}$, and $\underline{C_3}$ may be found.

The position of the maximum radial stress is found by equating the differential of equation (30) to zero, thus remembering that

$$m_1 = 2$$

$$\sigma_r = 2C_1 r^0 + C_2 m_2 r^{m_2 - 2} + C_3 m_3 r^{m_3 - 2} \tag{41}$$

$$\frac{d\sigma_r}{dr} = C_2 m_2 (m_2 - 2) r^{m_2 - 3} + C_3 m_3 (m_3 - 2) r^{m_3 - 3} = 0$$

thus

$$C_2 m_2 (m_2 - 1) r^{m_2 - 3} = - C_3 m_3 (m_3 - 2) r^{m_3 - 3}$$

$$r^{m_2 - m_3} = \frac{C_3 m_3 (m_3 - 2)}{C_2 m_2 (m_2 - 2)} \tag{42}$$

$$r = \left[\frac{C_3 m_3 (m_3 - 2)}{C_2 m_2 (m_2 - 2)}\right]^{\frac{1}{m_2 - m_3}}$$

Substituting (42) in (41)

$$\sigma_r \max = 2C_1 + C_2 m_2 \left[\frac{C_3 m_3 (m_3 - 2)}{C_2 m_2 (m_2 - 2)}\right]^{\frac{m_2 - 2}{m_2 - m_3}}$$

$$+ C_3 m_3 \left[\frac{C_3 m^3 (m_3 - 2)}{C_2 m^2 (m_2 - 2)}\right]^{\frac{m_3 - 2}{m_2 - m_3}} \tag{43}$$

According to equation (11) the stress in the direction of the length of the beam is found by taking the second derivation of equation (24), thus:

$$\sigma_\theta = \frac{\partial^2 \phi}{\partial r^2} = C_0 m_0 (m_0 - 1) r^{m_0 - 2} + C_1 m_1 (m_1 - 1) r^{m_1 - 2}$$

$$+ C_2 m_2 (m_2 - 1) r^{m_2 - 2)} + C_3 m_3 (m_3 - 1) r^{m_3 - 2} \qquad (44)$$

Remembering that $m_0 = 0$ and $m_1 = 2$

$$\sigma_\theta = 2C_1 + C_2 m_2 (m_2 - 1) r^{m_2 - 2)} + C_3 (m_3 - 1) r^{m_3 - 2} \qquad (45)$$

Presentation of Results

Using the simultaneous equations (38), (39), and (40) and the values of M given by equation (28), values for $\underline{C_1}$, $\underline{C_2}$, and $\underline{C_3}$ were calculated for a beam made of plainsawed boards.

For a beam made of quartersawed boards, equation (27) becomes:

$$\frac{\gamma}{\alpha} = \frac{E_L}{E_T} \; \frac{1 - \mu_{TR}\,\mu_{RT}}{1 - \mu_{LR}\,\mu_{TL}} = 17.2643 \qquad (46)$$

and equations (28) become

$$\begin{aligned} m_0 &= 0 \\ m_1 &= 2 \\ m_2 &= 5.155 \\ m_3 &= -3.155 \end{aligned} \qquad (47)$$

Slightly different values of $\underline{C_1}$ $\underline{C_2}$, and $\underline{C_3}$ are obtained

The longitudinal stress at both the concave and convex surface of the beam was obtained by use of equation (45). This stress was divided by the stress obtained by the usual engineering method ($\frac{MV}{I}$) for a straight beam to obtain a correction facto F for converting the engineering stress to the correct stress for the curved beams. These factors are plotted in figure 2 against the parameter

$$\frac{R + \frac{d}{2}}{R - \frac{d}{2}} \tag{48}$$

Each point in figure 2 represents six calculations, using six values of d 6, 12, 18, 24, 30, and 36. These six values of F for each value of $\frac{R + \frac{d}{2}}{R - \frac{d}{2}}$ was identical to five digits.

The maximum values of the radial stresses for beams made of plainsawed and quartersawed boards of Douglas-fir were calculated by means of equation (43). The values for beams made of plainsawed boards are plotted in figure 3 against $\frac{1}{R}$ and in figure 4 against $\frac{1}{d}$. Figures 5 and 6 are similar plots for beams made of quartersawed boards. All these figures show linear relationships. It can be deduced that the maximum radial stress is given by

$$\sigma_{r_{max}} = k \frac{M}{wRd} \tag{49}$$

where w is the width of the beam and M is the applied bending moment as shown in figure 1. It is readily shown by comparison with figures 3, 4, 5, and 6, that $k = \frac{3}{2}$ in agreement with the formula given by T. R. C. Wilson. [5]

It is noteworthy that the maximum radial stress does not occur where the tangential stress is zero; thus the radial strengthwouldbe reduced by the presence of the tangential stress according to Forest Products Laboratory Report 1816. [6]

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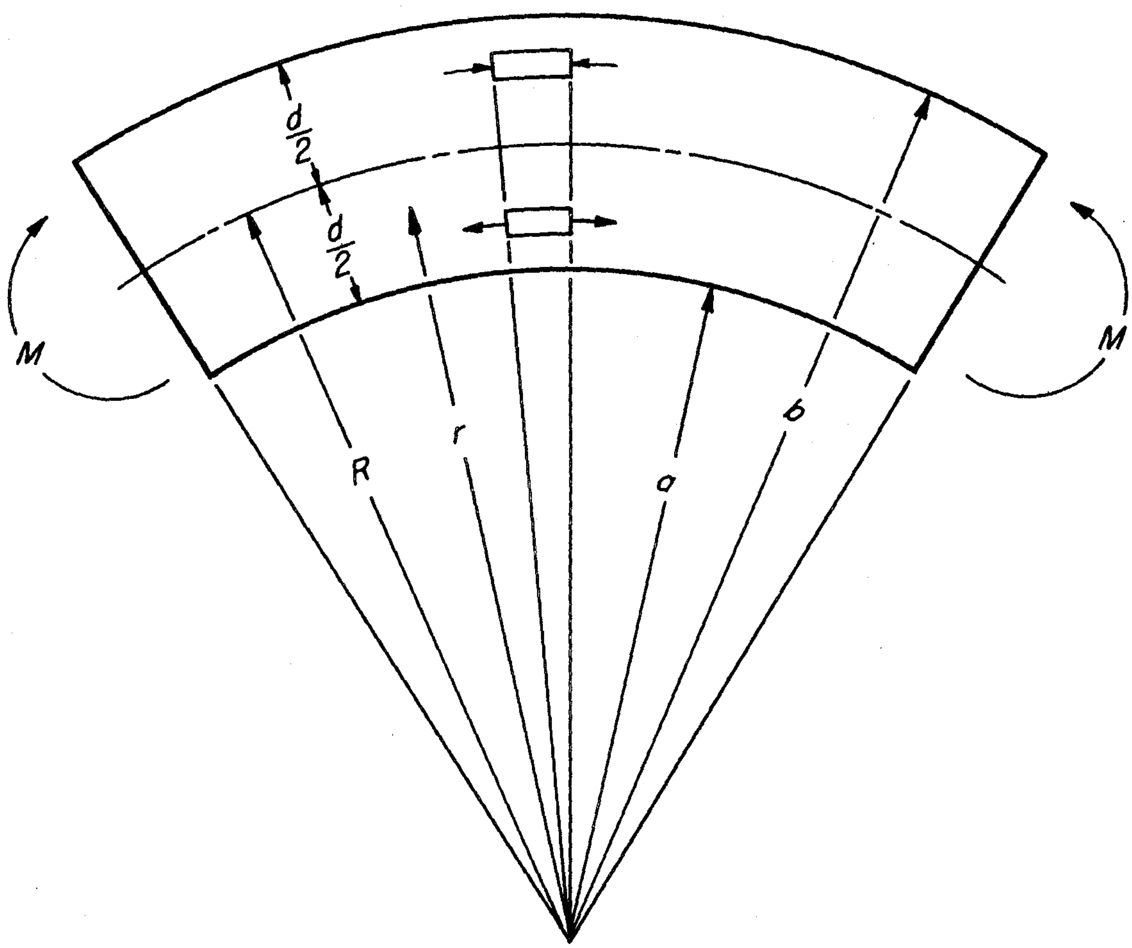
[5] Wilson, T. R. C. The Glued Laminated Wooden Arch. U.S. Department of Agriculture Technical Bulletin No. 691. p. 119. 1939.

[6] Norris, Charles B. Strength of Orthotropic Materials Subjected to Combined Stresses. U.S. Forest Products Lab. Report No. 1816. 40 p. 1950.



M 125 330

Figure 1.--Sketch of curved beam, giving nomenclature used.

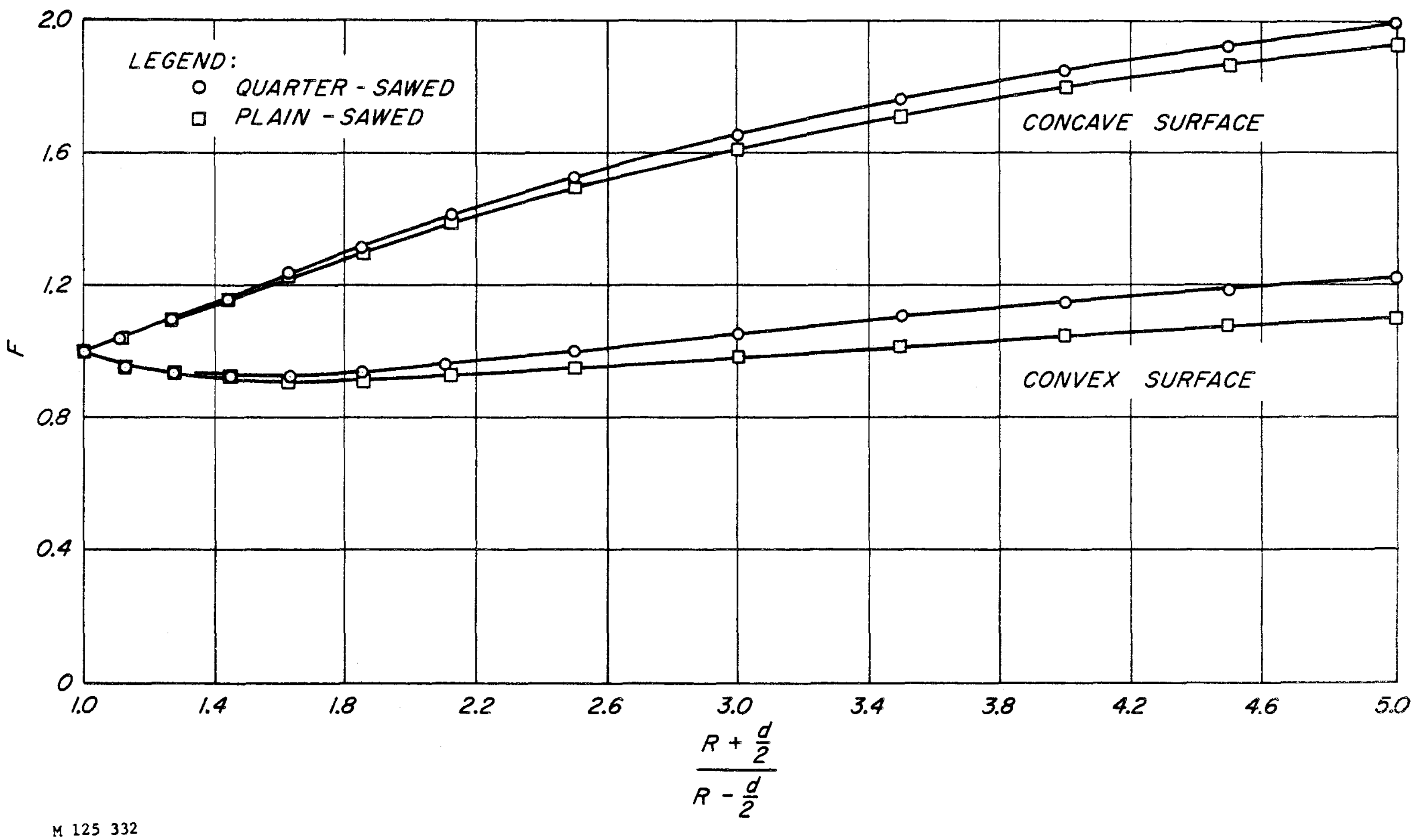
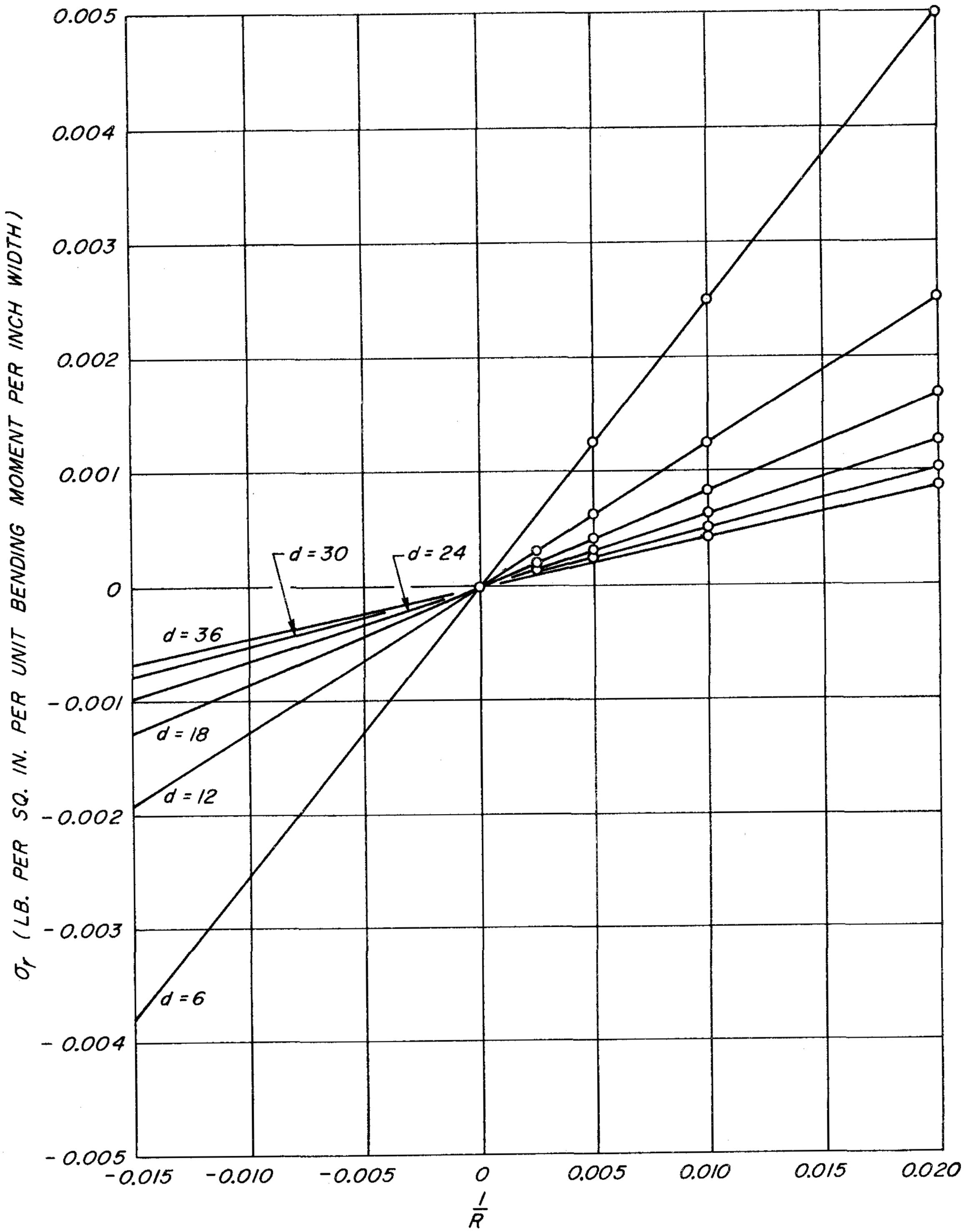


Figure 2.--Correction factors for curved beams made of plainsawed and quartersawed Douglas-fir.



M 125 335

Figure 3.--Maximum radial stress in a beam made of plainsawed boards, plotted against curvature of beam.

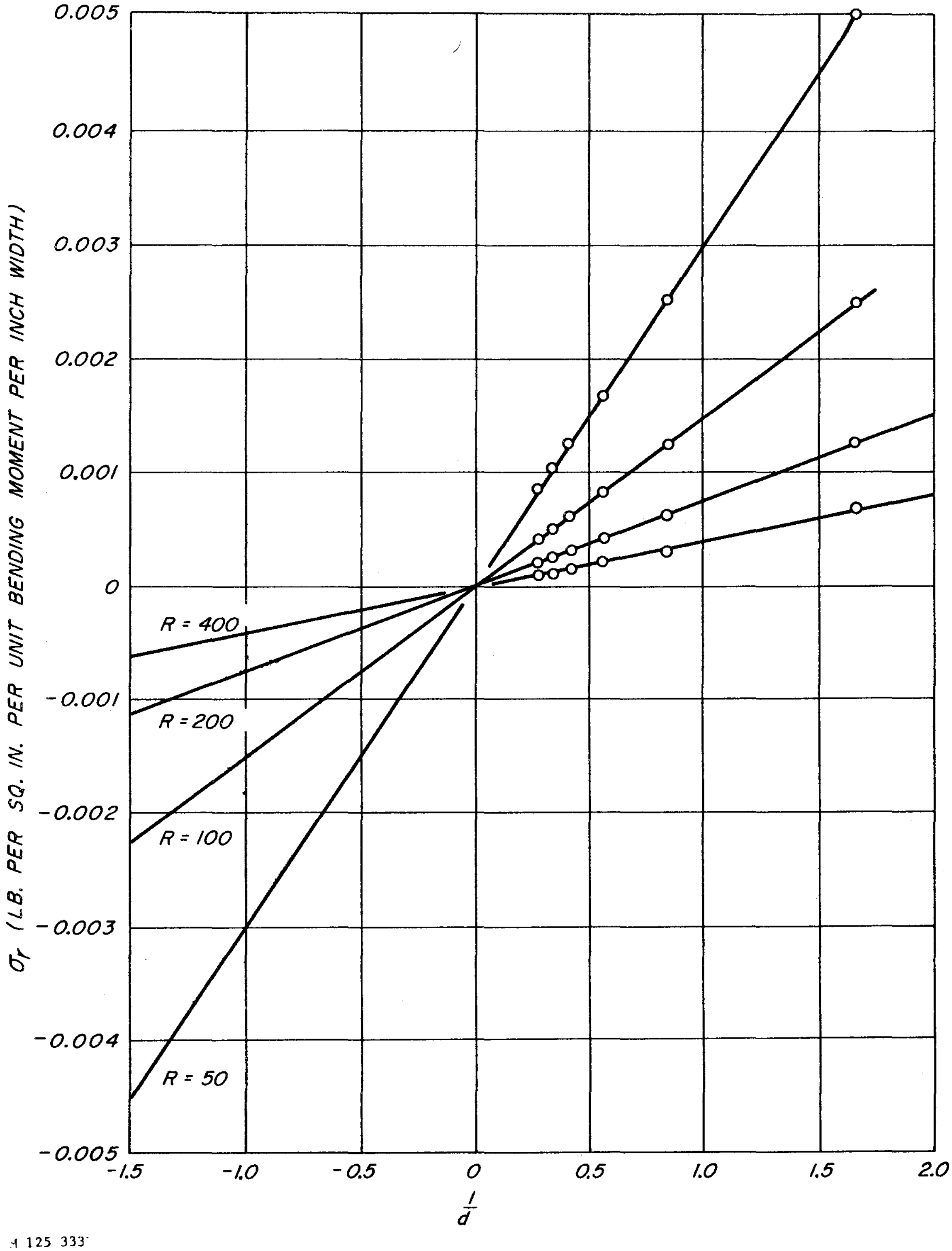
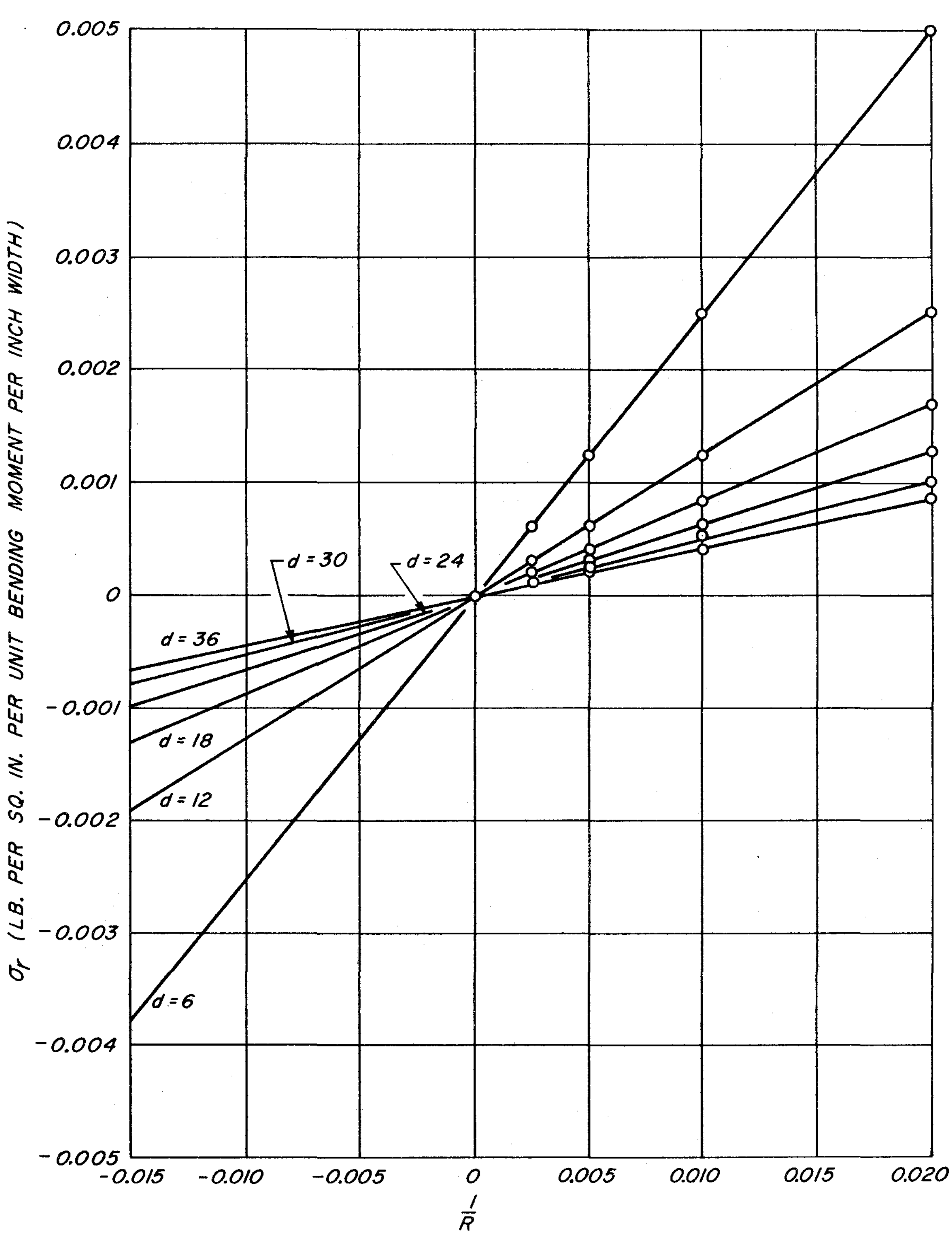
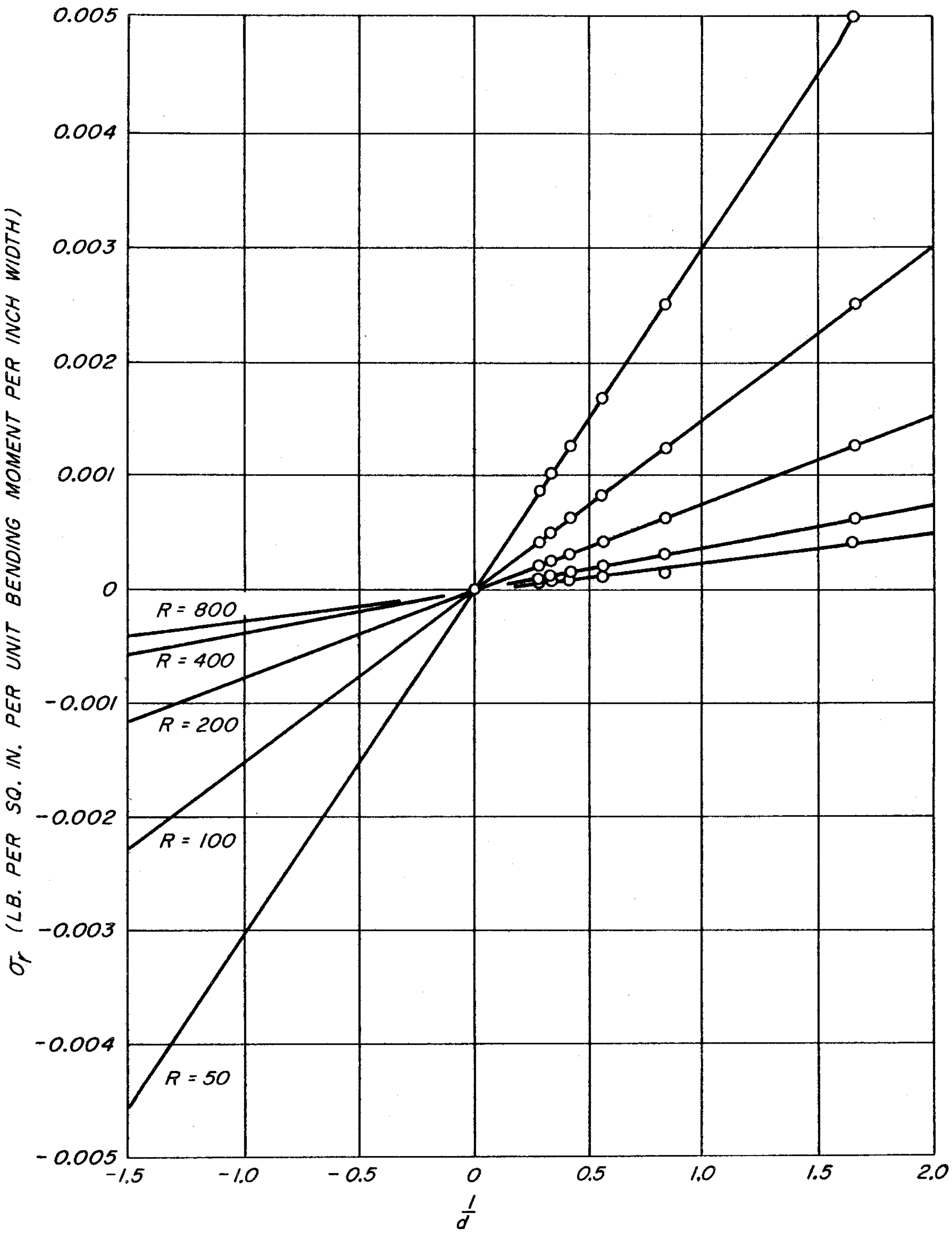


Figure 4.--Maximum radial stress in abeam made of plainsawed boards, plotted against reciprocal of beam depth.



M 125 331

Figure 5.--Maximum radial stress in a beam made of quartersawed boards, plotted against curvature of beam.



M 125 334

Figure 6.--Maximum radial stress in a beam made of quartersawed boards, plotted against reciprocal of beam depth.